QUASICONVEXITY OF BANDS IN HYPERBOLIC 3-MANIFOLDS

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Abstract. Let M by a complete hyperbolic 3-manifold homotopy equivalent to a compact surface Σ. Let Φ be a proper subsurface of Σ, whose boundary is sufficiently short in M. We show that the union of all Margulis tubes and cusps homotopic into Φ lifts to a uniformly quasiconvex subset of hyperbolic 3-space.

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1. Introduction

Let Σ be a compact orientable surface of genus g and with p boundary components. We assume that the "complexity", $\xi(\Sigma) = 3g + p - 3$, of Σ is strictly positive. Suppose that $\Sigma = \pi_1(\Sigma)$ acts properly discontinuously on hyperbolic 3-space, $\mathbb{H}^3$, by orientation preserving isometries. The quotient $\mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold with a natural homotopy equivalence to Σ. We will assume (for now) that the cusps of M are in bijective correspondence with the boundary components of Σ (that is, the action of Γ is "strictly type preserving"). Let $\eta > 0$ be some constant less than the 3-dimensional Margulis constant. Let $\Psi(M, \eta)$ be the closed non-cuspidal part of M, that is M minus union of open η-Margulis cusps. By tameness [Bon], $\Psi(M, \eta)$ is homeomorphic to $\Sigma \times \mathbb{R}$. The proof of the Ending Lamination Conjecture [Mi, BrCM] has lead to a reasonably good understanding of such manifolds in terms of model spaces. An important feature of their geometry are "bands" — subsets homeomorphic to a subsurface of Σ times an interval, where the boundary curves of the subsurface are represented by short geodesics in M (see for example, [Mi, BrCM, Mj1, Mj2, Bow1, Bow4]). Subsets of this sort are termed "blocks" in [Mj1, Mj2], though in this paper, we use the terminology from [Bow1] to avoid a clash with the term "block" as used in [Mi]. Bands are related to the "scaffolds" featuring in [BrCM]. The aim of this paper is to show that a lift of any such band (together with the associated Margulis tubes and cusps) is uniformly

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quasiconvex in $\mathbb{H}^3$. By "uniform" we mean that the quasiconvexity constant depends only on $\xi(\Sigma)$.

(For the purposes of this introduction, one can imagine the case where $\partial\Sigma = \emptyset$, so that there are no cusps. In this case, $M \cong \Sigma \times \mathbb{R}$. One can also think of the geometrically finite case, where tameness is elementary.)

Here is a more precise statement. Let $\Phi \subseteq \Sigma$ be a connected proper subsurface of positive complexity. We assume that there are no disc components in its complement, so that we can identify $G = \pi_1(\Phi)$ as a subgroup of $\Gamma = \pi_1(M)$. (Note $\xi(\Phi) \leq \xi(\Sigma)$.) Let $\tilde{\Phi}$ be a lift of Φ to the universal cover, $\tilde{\Sigma}$, of Σ, so that $\Phi = \tilde{\Phi}/G$. Given any $\eta > 0$, let $\tilde{T}(\tilde{\Phi}, \eta)$ be the set of points of $\tilde{M}$ which are displaced a distance at most η by some non-trivial element of G. Recall that a subset $Q \subseteq \mathbb{H}^3$ is r-*quasiconvex* if, for all $x, y \in Q$, the geodesic segment $[x, y] \subseteq \mathbb{H}^3$ lies in the r-neighbourhood, $N(Q, r)$ of Q. We show:

Main Theorem. *Let Σ be a compact orientable surface, let Φ be a proper subsurface, and let M be a product manifold, as above. We can choose η sufficiently small depending only on $\xi(\Sigma)$ so that the following holds. Suppose that each component of $\partial\Phi$ is realised by a closed curve of length less than η in M. Then $\tilde{T}(\tilde{\Phi}, \eta)$ is r-quasiconvex in $\mathbb{H}^3$, where r depends only on $\xi(\Sigma)$ (and our choice of η).*

We say such sets are "uniformly quasiconvex" — we can choose η and hence r so as to depend only on $\xi(\Sigma)$. The statement is quite robust. (For example, if $t \geq \eta$ is any number greater than η, then $\tilde{T}(\tilde{\Phi}, t)$ lies in some bounded neighbourhood of $\tilde{T}(\tilde{\Phi}, \eta)$, and is thus r'-quasiconvex, where r' might also depend on t. This is a consequence of the argument in Section 4, cf. [Bow3], though we shall not give details here.)

To set the result in context, here is another way of describing these sets. Let $T_M(\eta)$ be the "η-thin" part of M, i.e. the set of points of M of injectivity radius at most $\eta/2$. If η is less than the Margulis constant, then $T_M(\eta)$ is a disjoint union of Margulis tubes and cusps. Moreover, following Otal [Ot], if we assume that η is small enough depending only on $\xi(\Sigma)$, then the core curve of any Margulis tube is homotopic to a simple closed curve in Σ (under the above homotopy equivalence). Moreover, the set of all such tubes is topologically unlinked in $M \cong \Psi(M, \eta) \times \mathbb{R}$. These statements are proven in [Ot]. Let $T(\Phi, \eta) \subseteq T_M(\eta)$ be the union of all those η-Margulis tubes and cusps which can be homotoped into Φ. Then, $T(\Phi, \eta) = \tilde{T}(\tilde{\Phi}, \eta)/G$, so we can think of $\tilde{T}(\tilde{\Phi}, \eta)$ as the "lift" of $T(\Phi, \eta)$ corresponding to $\tilde{\Phi}$.

Loosely speaking, a "band" (as defined in [Bow1]) in $\Psi(M, \eta)$ is the image of $\Phi \times [-1, 1]$ under a suitable homeomorphism of $\Psi(M, \eta)$ with $\Sigma \times \mathbb{R}$. Its "vertical boundary", $\partial\Phi \times [-1, 1]$ lies in the corresponding set of Margulis tubes and cusps. Such a band arises whenever the end invariants of M have large subsurface projection distance in Φ, see [Mi, BrCM, Bow2]. They behave intrinsically like product manifolds of lower complexity. In [OhS], subsets of this sort feature in their account of geometric limits of product manifolds. They are also used in [Bow2], and some of the papers mentioned therein. Some combinatorial description of bands is given in [Bow1], but much remains to be understood about the geometry of how such bands lie inside M. Our result tells us that if we include in the band all the Margulis tubes and cusps that it meets, then it lifts to a uniformly quasiconvex subset of $\mathbb{H}^3$ (since such a lift lies in a uniform neighbourhood of $\tilde{T}(\tilde{\Phi}, \eta)$).

A particulary significant application of bands of this sort can be found in the work of Mj towards showing that limit sets of geometrically finite kleinian groups are locally connected if they are connected [Mj1, Mj2]. In particular, in [Mj1] it is shown that sets of the form $\tilde{T}(\tilde{\Phi}, \eta)$ (as feature in the Main Theorem) are quasiconvex, though without any uniformity statement — the quasiconvexity constant, r, might depend on the particular band. It is hoped that the result presented here might offer new insights into these constructions, and we aim to explore these ideas further.

For ease of exposition, we will split our statement in two. Theorem 2.1 gives a formal statement of the result when $\partial\Sigma = \emptyset$. We will prove this first, in Sections 4 to 6. Theorem 2.2 gives a formal statement in general, and we explain the modifications necessary to prove it in Section 7. We also explain there how do deal with the "type preserving" case where there might be accidental parabolics.

2. Statement of results

Let Σ be a compact orientable surface of complexity $\xi(\Sigma) = 3g + p - 3 > 0$, as defined in the introduction. We write $\Sigma = \tilde{\Sigma}/\Gamma$, where $\Gamma = \pi_1(\Sigma)$. Let $X(\Sigma)$ be the set of free homotopy classes of non-peripheral simple closed curves in Σ. (This is the vertex set of Harvey's curve complex.)

Definition. A *proper subsurface* of Σ is a compact subsurface, $\Phi \neq \Sigma$, which is not a disc or an annulus and such that no component of the complement is homotopically trivial or homotopic into $\partial\Sigma$.

In other words, we are ruling out disc and peripheral annular components in the complement of Φ. (We are, however, allowing for non-peripheral annular components of the complement.)

We can write $\partial\Phi = \partial_\Sigma\Phi \sqcup \partial_C\Phi$, where $\partial_C\Phi = \partial\Phi \cap \partial\Sigma$, and where $\partial_\Sigma\Phi = \partial\Phi \setminus \partial\Sigma$ is the relative (toplogical) boundary of Φ in Σ. We will generally consider Φ as defined up to homotopy in Σ, which can be assumed to fix $\partial_C\Phi$. Note that $\xi(\Phi) \leq \xi(\Sigma)$. We write $X(\Sigma, \Phi) \subseteq X(\Sigma)$ for the set of curves in Σ which can be homotoped into Φ. Thus $X(\Sigma, \Phi) = X(\Phi) \sqcup X(\Sigma, \partial_\Sigma\Phi)$, where $X(\Phi)$ is defined intrinsically to Φ, and where $X(\Sigma, \partial_\Sigma\Phi)$ is the set of homotopy classes of components of $\partial_\Sigma\Phi$. (Note that two components of $\partial_\Sigma\Phi$ get identified if they bound an annulus in $\Sigma \setminus \Phi$.)

We choose a lift, $\tilde{\Phi}$, of Φ to $\tilde{\Sigma}$. We can write $\Phi = \tilde{\Phi}/G$, where $G \leq \Gamma$ is a subgroup naturally isomorphic to $\pi_1(\Phi)$. (Note that $\pi_1(\Phi)$ injects into $\pi_1(\Sigma)$.)

Let us first consider the case where $\partial\Sigma = \emptyset$. Suppose that $\Gamma = \pi_1(\Sigma)$ acts properly discontinuously by orientation preserving isometries, without parabolics, on $\mathbb{H}^3$. By tameness [Bon], the quotient $M = \mathbb{H}^3/\Gamma$ is homeomorphic to $\Sigma \times \mathbb{R}$. Given $\eta > 0$, let $\tilde{T}(\tilde{\Sigma}, \eta)$ be the set of points displaced a distance at most η by some non-trivial element of Γ. Let $T_M(\Sigma, \eta) = \tilde{T}(\tilde{\Sigma}, \eta)/\Gamma$. This is the η-thin part of M. If η is less than the 3-dimensional Margulis constant, then each component, T, of $T_M(\Sigma, \eta)$ is a uniform neighbourhood of some embedded closed geodesic, α_M, in M. We will assume this to be a solid torus. (It is possible that $T = \alpha_M$, but this makes no essential difference to the argument, and we can simply discard such components anyway.) We refer to T as a *Margulis tube*. Each lift of T to $\mathbb{H}^3$ is a uniform neighbourhood of a bi-infinite geodesic.

In fact, it is shown in [Ot] that if η is less than some constant, say $\eta_0(g) > 0$, depending only on $g = \mathrm{genus}(\Sigma)$, then α_M is homotopic to a simple closed curve in Σ (under the natural homotopy equivalence of $M \cong \Sigma \times \mathbb{R}$ with Σ). Given $\alpha \in X(\Sigma)$, write $\alpha_M \subseteq M$ for its geodesic realisation, in M, and let $l_M(\alpha) = \mathrm{length}(\alpha_M)$. We write $X(M, \eta) = \{\alpha \in X(\Sigma) \mid l_M(\alpha) \leq \eta\}$. Then by [Ot], we see that $\eta \leq \eta_0(g)$, then $T_M(\tilde{\Sigma}, \eta) = \bigsqcup_{\alpha \in X(M,\eta)} T_M(\alpha, \eta)$, where $T_M(\alpha, \eta)$ is the Margulis tube with core curve α_M. In fact, [Ot] tells us that the set of such tubes (or equivalently their core curves) is unlinked in $\Sigma \times \mathbb{R}$. (That is, there is a map $t : X(M, \eta) \longrightarrow \mathbb{R}$, and a homeomorphism of M with $\Sigma \times \mathbb{R}$, such that each α_M gets sent into $\Sigma \times \{t(\alpha)\}$.)

Let $\Phi \subseteq \Sigma$ be a proper surface and $\tilde{\Phi}$ is a lift to $\tilde{\Sigma}$. Let $T(\Phi, \eta) = T_M(\Phi, \eta) = \tilde{T}(\tilde{\Phi}, \eta)/G \subseteq T_M(\Sigma, \eta)$ be the η-thin part corresponding to

Φ, as defined in Section 1. Thus, $T(\Phi,\eta) = \bigsqcup_{\alpha \in X(\Sigma,\Phi)\cap X(M,\eta)} T_M(\alpha,\eta) \subseteq T_M(\Sigma,\eta)$. We will suppose that $X(\Sigma,\partial\Phi) \subseteq X(M,\eta)$, that, is all the boundary curves of Φ are short in M. We will show:

Theorem 2.1. $(\forall g \in \mathbb{N})(\exists \eta(g) > 0)(\forall \eta \in (0,\eta(g)])(\exists r \geq 0)$ *with the following property. Suppose that Σ is a closed orientable surface of genus g and that $\Phi \subseteq \Sigma$ is a proper subsurface. Let $\tilde{\Phi} \subseteq \tilde{\Sigma}$ be a component of the preimage of Φ in $\mathcal{T}$. Suppose that $\Gamma = \pi_1(\Sigma)$ acts properly discontinuously by orientation preserving isometries without parabolics on $\mathbb{H}^3$, with quotient $M = \mathbb{H}^3/\Gamma$. Suppose that $X(\Sigma,\partial\Phi) \subseteq X(\Sigma,\eta)$. Then $\tilde{T}(\tilde{\Phi},\eta)$ is r-quasiconvex in $\mathbb{H}^3$.*

As noted in the introduction, the hypotheses on Φ are natural. If $X(M,\eta) \neq \emptyset$, then there will always be such a surface. In fact, generically there will be many such — they arise whenever we have a large subsurface projection distance between the two end invariants of M. (See [Mi, BrCM, Bow2] etc.)

Suppose now that Σ is any compact surface. We again suppose that $\pi_1(\Sigma)$ acts properly discontinuously on $\mathbb{H}^3$ with quotient $M = \mathbb{H}^3/\Gamma$. We now assume that the cusps coincide precisely with the boundary curves of Σ (i.e. the action is "strictly type preserving"). Associated to each boundary curve, $\alpha \subseteq \partial_C\Phi$, we have an η-Margulis cusp, $P_M(\alpha,\eta) \subseteq M$. Provided η is less than the Margulis constant, these are disjoint. Let $\Psi(M,\eta) = M \setminus \bigcup_\alpha \operatorname{int} P_M(\alpha,\eta)$ as α ranges over the boundary components. This "non-cuspidal" part of M is homeomorphic to $\Sigma \times \mathbb{R}$ [Bon]. If, in addition, η is sufficiently small depending on $\xi(\Sigma)$, then each Margulis tube has the form $T_M(\alpha,\eta)$ for $\alpha \in X(\Sigma)$, as before. Moreover the set of Margulis tubes is unlinked in $\Psi(M,\eta)$ (see [Ot]).

Let $\Phi \subseteq \Sigma$ be a proper subsurface. We write $T(\Phi,\eta) = \bigcup\{P_M(\alpha,\eta) \mid \alpha \subseteq \partial\Sigma \cap \partial\Phi\} \cup \bigcup\{T_M(\alpha,\eta) \mid \alpha \in X(\Sigma,\Phi)\}$. We similarly define $\tilde{T}(\tilde{\Phi},\eta)$ where $\tilde{\Phi}$ is a lift of Φ to $\tilde{\Sigma}$. In this case we have the following generalisation of Theorem 2.1:

Theorem 2.2. $(\forall g,p \in \mathbb{N})(\exists \eta(g,p) > 0)(\forall \eta \leq \eta(g,p))(\exists r \geq 0)$ *with the following property. Suppose that Σ is a compact orientable surface of genus g with p boundary components, and that $\Phi \subseteq \Sigma$ is a proper subsurface. Let $\tilde{\Phi} \subseteq \tilde{\Sigma}$ be a component of the preimage of Φ in $\tilde{\Sigma}$. Suppose that $\Gamma = \pi_1(\Sigma)$ has a strictly type preserving properly discontinuous action on $\mathbb{H}^3$ by orientation preserving isometries. Let $M = \mathbb{H}^3/\Gamma$. Suppose that $X(\Sigma,\partial_\Sigma\Phi) \subseteq X(\Sigma,\eta)$. Then $\tilde{T}(\tilde{\Phi},\eta)$ is r-quasiconvex in $\mathbb{H}^3$.*

We can also generalise to allow for "accidental parabolics". In this case, we suppose that each boundary of Σ corresponds to a $\mathbb{Z}$-cusp, but not necessarily conversely. It necessarily holds that each non-peripheral cusp in M will be homotopic to a simple closed curve in $X(\Sigma)$. In defining $T(\Phi, \eta)$ we include these in place of Margulis tubes, whenever the curve is homotopic into Φ. The result then goes through without change.

We will discuss these generalisations in Section 7.

3. Convex hulls

We reduce the Main Theorem to a statement about convex hulls. Let Σ, M, Φ, $\tilde{\Phi}$ etc. be as in the statements of Theorem 2.1 or 2.2. We fix some $\eta \leq \eta(g)$. Let $G \leq \Gamma$ be the setwise stabiliser of $\tilde{\Phi}$. Thus $G \cong \pi_1(\Phi)$ and $\Phi = \tilde{\Phi}/G$. Let ΛG be the limit set of the restricted action of G on $\mathbb{H}^3$, and write $\tilde{H}$ be the convex hull of ΛG in $\mathbb{H}^3$. Thus, $H = \tilde{H}/G$ is the convex core of $\mathbb{H}^3/G$, that is the smallest closed subset with locally convex boundary whose inclusion into $\mathbb{H}^3/G$ is a homotopy equivalence.

Note that any closed geodesic, α_M in M lies in H. In particular, any Margulis tube T will meet H. Each lift, $\tilde{T}$, to $\mathbb{H}^3$ is convex and meets $\tilde{H}$. We deduce:

Lemma 3.1. *$\tilde{H} \cup \tilde{T}(\tilde{\Phi}, \eta)$ is r_0-quasiconvex in $\mathbb{H}^3$ for some fixed $r_0 \geq 0$.*

Proof. It's enough to note that any two points of $\tilde{H} \cup \tilde{T}(\tilde{\Phi}, \eta)$ are connected by a path in $\tilde{H} \cup \tilde{T}(\tilde{\Phi}, \eta)$ consisting of at most three geodesic segments. □

Therefore, the main thing we need to show is:

Proposition 3.2. *If Φ is as above, then $\tilde{H} \subseteq N(\tilde{T}(\tilde{\Phi}, \eta), s)$, where s depends only on $\xi(\Sigma)$ and on η.*

In the closed surface case, Proposition 3.2 will be a consequence of Propositions 4.1 and 6.12 combined. This will be dealt with in Sections 4–6. We discuss the general case in 7.

Given Proposition 3.2, we can prove Theorems 2.1 and 2.2:

Proof of Theorems 2.1 and 2.2. Given $x, y \in \tilde{T}(\tilde{\Phi}, \eta)$, by Lemma 3.1, $[x, y] \subseteq N(\tilde{H} \cup \tilde{T}(\tilde{\Phi}, \eta), r_0)$ and so by Proposition 3.2, $[x, y] \subseteq N(\tilde{T}(\tilde{\Phi}, \eta), s + r_0)$, so $\tilde{T}(\tilde{\Phi}, \eta)$ is $(s + r_0)$-quasiconvex in $\mathbb{H}^3$. □

We now set about proving Proposition 3.2.

4. Geometry of handlebodies

For Sections 4 to 6 we will assume that Σ is a closed surface of genus g, and that $\Gamma = \pi_1(\Sigma)$ acts on $\mathbb{H}^3$ without parabolics.

Let $\Phi = \tilde{\Phi}/G$ be a proper subsurface, where $\tilde{\Phi} \subseteq \tilde{\Sigma}$ and $G \equiv \pi_1(\Phi)$. Let $X(\partial\Phi)$ be the set of boundary curves of Φ (thought of as defined up to homotopy in Φ), and let $X_0(\Phi) = X(\Phi) \sqcup X(\partial\Phi)$. The inclusion $X(\Phi) \hookrightarrow X(\Sigma, \Phi)$ extends to a natural map $X_0(\Phi) \longrightarrow X(\Sigma, \Phi)$, which identifies any pair of curves in $X(\partial\Phi)$ which bound an annulus in $\Sigma \setminus \Phi$.

Let $V = \mathbb{H}^3/G$, and let $H = \tilde{H}/G$ be the convex core of V. We write $\pi_V : \mathbb{H}^3 \longrightarrow V$, $\pi_M : \mathbb{H}^3 \longrightarrow M$, and $\pi_{VM} : V \longrightarrow M$ for the covering maps, so that $\pi_M = \pi_{VM} \circ \pi_V$.

Recall that $\tilde{T}(\tilde{\Phi}, \eta)$ is a G-invariant subset of $\mathbb{H}^3$. We write $T_V(\Phi, \eta) = \pi_V(\tilde{T}(\tilde{\Phi}), \eta) \subseteq V$. Thus $T_V(\Phi, \eta) = \bigsqcup\{T_V(\alpha, \eta) \mid \alpha \in X_0(\Phi)\}$, where $T_V(\alpha, \eta)$ is the Margulis tube with core curve α_V in V. Note that $\pi_{VM}(T_V(\Phi, \eta)) = T_M(\Phi, \eta) \subseteq M$. Here $\pi_{VM}|T_V(\Phi, \eta)$ is injective, except that it might identify pairs of tubes that correspond to boundary curves of Φ bounding annuli in Σ.

We assume that V is topologically finite. (In fact, this is necessarily the case by tameness — see Section 6.) Since we are assuming there are no parabolics, this implies that H is compact. Since $\pi_1(H) \cong G$ is free, H must be a handlebody [He]. The boundary, $S = \partial H$, is a closed surface, whose induced path metric is locally hyperbolic (see [T] or Section 1.12 of [EM]). By comparing genera, we see that ∂H is homeomorphic to the double, $D\Phi$, of Φ. In particular we can write $S = \Phi^- \cup \Phi^+$ where Φ^- and Φ^+ are each homeomorphic to Φ and $(\partial\Phi)_S = \Phi^- \cap \Phi^+$ is a disjoint union of closed curves. A-priori, there are many ways of cutting S into two such subsurfaces, but, as we discuss in Sections 5 and 6, there is a preferred choice of homotopy class of $(\partial\Phi)_S$ in S, and the homeomorphisms of Φ with $\Phi^\pm$ can be taken to lie in the natural homotopy class of the equivalence of Φ with V. We will assume that the components of $(\partial\Phi)_S$ are intrinsically geodesic in S. (We should qualify the above by noting that there is a "degenerate" fuchsian case, where H is a totally geodesic surface homeomorphic to Φ. Then $S = \Phi = \Phi^- = \Phi^+$. The relevant constructions are readily reinterpreted, though the result, in this case, is elementary.)

The main result of this section is the following.

Proposition 4.1. *Suppose that $G = \pi_1(\Phi)$ acts properly discontinously on $\mathbb{H}^3$ with quotient V. Suppose that the convex core, H, of V is compact. Suppose that, for each boundary curve, α, of Φ, we have a simple closed geodesic, α_S, in ∂H, with α_S, homotopic to α in V (under* *the natural homotopy equivalence of V with Φ). Suppose that we can write $\partial H = \Phi^- \cup \Phi^+$, where $\Phi^\pm$ are homeomorphic to Φ, and where $\Phi^- \cap \Phi^+$ is the union of the α_S. Suppose that for some $l \geq 0$, the length of each α_S is at most l. We also assume that each α_S is homotopic to a curve of length at most η in V. Let $T_V(\Phi, \eta)$ be the union of η-Margulis tubes, whose core curves are homotopic in V to simple closed curves in Φ (again under the natural homotopy equivalence). Then $H \subseteq N(T_V(\Phi, \eta), s)$, where s depends only on l, η and $\xi(\Phi)$.*

Under the hypotheses of Theorem 2.1, we will see (Lemma 6.3) that $T_V(\Phi, \eta)$ accounts for all of the thin part of V. We will also see, under the hypotheses of Theorem 2.1, that l can be bounded in terms of η and g (Proposition 6.12). Thus s will ultimately depend only on these, thereby proving Proposition 3.2.

The idea of the proof of Proposition 4.1 is to construct a homotopy from Φ^- to Φ^+ in V fixing $(\partial\Phi)_S$ and where the image lies in a uniform neighbourhood of $T_V(\Phi, \eta)$. Thought of as a 3-dimensional homology chain, the homotopy must map with degree 1 to H. In particular, its image contains H. In practice, we first construct a homotopy from $(\partial\Phi)_S$ to $(\partial\Phi)_V = \bigcup\{\alpha_V \mid \alpha \in X_0(\Phi)\}$ and then carry out a homotopy fixing $(\partial\Phi)_V$. The construction uses well known ideas from the interpolation of pleated surfaces etc., so we only sketch the argument. A similar construction is used in [Bow3].

In what follows, 3HS, 4HS, 1HT refer to the "three-holed sphere", "four-holed sphere" and "one-holed torus".

By a "complete" multicurve, γ, in Φ, we mean the realisation of a maximal collection of disjoint curves in $X_0(\Phi)$. Thus, $\partial\Phi \subseteq \gamma$, and each component of $\Phi \setminus \gamma$ is a 3HS. (In other words, $\gamma \setminus \partial\Phi$ is a pants decomposition.) The following expresses the fact that the pants graph is connected [HaT].

Lemma 4.2. *Suppose γ, δ are complete multicurves. Then there is a sequence, $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_n = \delta$, of complete multicurves such that for each $i = 0, \ldots, n-1$, there are curves, α in γ_i and β in γ_{i+1}, such that $\gamma_i \setminus \alpha = \gamma_{i+1} \setminus \beta$, and such that $\alpha \cup \beta$ has a regular neighbourhood that is either a 1HT or a 4HS.*

It follows that we can take such a regular neighbourhood of $\alpha \cup \beta$ to be a component of $\Phi \setminus (\gamma_i \setminus \alpha) = \Phi \setminus (\gamma_{i+1} \setminus \beta)$ and that α and β intersect exactly once or twice respectively. Note that Lemma 4.2 is vacuous if Φ is itself a 3HS, in which case $X(\Phi) = \emptyset$.

Before continuing, we note the following well known observation:

Lemma 4.3. *Let F be any hyperbolic surface with (possibly empty) boundary. Then each component of the η-thick part of Σ has diameter bounded above in terms of $\xi(F)$ and η.*

Recall that the "η-thick part" of Σ is the set of points of injectivity radius at least $\eta/2$. Thus Lemma 4.3 is a simple consequence of the fact that the area of F is bounded, and so there is a bound on the number of $\eta/2$-balls one can pack disjointly into the thick part F.

We next describe some fairly standard constructions. They are closely related to those described in Section 2 of [Bow3].

(C1): (cf. Section 1.3 of [Bon]). Suppose that $\gamma \subseteq \Phi$ is a complete multicurve. Then there is a hyperbolic structure on Φ, with geodesic boundary, $\partial\Phi$, and a 1-lipschitz map, $\phi : \Phi \longrightarrow V$ such that if $\alpha \subseteq \gamma$ is any component, then $\phi|\alpha$ maps α locally isometrically to the geodesic realisation, α_V, in V. We refer to ϕ as *realising* γ.

Note that if T is a Margulis tube with $\phi^{-1}(T)$ containing a non-trivial curve in Φ, then $T = T_V(\alpha, \eta)$ for some $\alpha \in X_0(\Phi)$. Moreover, each component of $\Phi \setminus \phi^{-1}(T_V(\Phi, \eta))$ has bounded diameter in terms of $\xi(\Sigma)$ and η. Since we are assuming that $\partial\Phi \subseteq T_V(\Phi, \eta)$, we see that $\phi(\Phi)$ lies in a bounded neighbourhood of $T_V(\Phi, \eta)$.

We are assuming that Φ is proper, but for future reference (in Section 5) we note that this construction also applies if $\Phi = \Sigma$. The definitions are the same: a complete multicurve is a union of curves cutting Σ into 3HS's.

We also make the following observations regarding homotopies:

(C2): Given $l \geq 0$ and $\eta > 0$, there is some $r \geq 0$ such that if α is any essential curve in V of length at most l, then we can find a homotopy of α to α_V in V whose image lies in $N(\alpha_V \cup T_V(\alpha, \eta), r)$.

(C3): Given $l \geq 0$, $\eta > 0$, there is some r depending only on l, η and $\xi(\Phi)$ with the following property. Suppose that $\phi : \Phi \longrightarrow V$ is a 1-lipschitz map with respect to a hyperbolic structure on Φ, in which each component of $\partial\Phi$ is intrinsically geodesic and of length at most l. Then there is a complete multicurve, γ, and a map $\phi' : \Phi \longrightarrow V$ of the type described by (C1), together with a homotopy from ϕ to ϕ' whose image lies in $N(T_V(\Phi, \eta), r)$. Moreover, we can assume that for any boundary curve α, $\phi|\alpha$ is any prescribed homotopy of the type described by (C2).

(C4): Suppose that γ and δ are complete multicurves that are equal, or differ by a move of the type described by Lemma 4.2. Suppose that ϕ and ϕ' are maps of the type described by (C1) with respect to γ and δ respectively. Then there is a homotopy from ϕ to ϕ' in V, fixing $\partial\Phi$ setwise, and whose image lies in $N(T_V(\Phi,\eta),r)$, where r depends only on η and $\xi(\Sigma)$. (Here the components of γ and δ can be arbitrarily long.)

The above constructions are fairly standard. In (C1), the map ϕ can be a pleated surface as described by Thurston, [T, Bon]. Note that the η-thin part of Φ maps to the η-thin part of V, and so each component of $\Phi \setminus \phi^{-1}(T_V(\Phi,\eta))$ has diameter bounded above by Lemma 4.3. For (C3), note that by the Bers Lemma, we can find a complete geodesic multicurve on Φ, whose length is bounded in terms of l and $\xi(\Phi) \leq \xi(\Sigma)$. We can now homotope these curves to their geodesic realisations in V using (C2). This reduces us to proving (C3) for 3HS's which is fairly straightforward. The construction of (C4) similary reduces to the case of a 3HS, or to elementary moves on a 1HT or 4HS.

We can now prove Proposition 4.1. A more detailed discussion of our constructions can be found in [Bow3]. See, in particular, Section 3 thereof.

Proof of Proposition 4.1. We have maps $\theta^\pm : \Phi \longrightarrow \Phi^\pm$, from Φ to $\Phi^\pm$, which are locally isometric, in particular 1-lipschitz with respect to hyperbolic structures induced on Φ. Each boundary component of Φ is geodesic in each of these structures and of length bounded by l.

Use (C2) above to homotope each component α of $\Phi^- \cap \Phi^+$ to the corresponding geodesic, α_V, in V. By (C3) we extend these to homotopies of $\theta^\pm$ to maps $\phi^\pm : \Phi \longrightarrow V$ of the type described by (C1), with respect to complete multicurves, $\gamma^\pm$.

Now let $\gamma^- = \gamma_0, \gamma_1, \ldots, \gamma_n = \gamma^+$ be a sequence of complete multicurves given by Lemma 4.2. Let $\phi_i : \Phi \longrightarrow V$ be a map of type (C1) which realises γ_i. We can assume that $\phi_0 = \phi^-$ and $\phi_n = \phi^+$. By (C4) we have a homotopy between ϕ_i and ϕ_{i+1}, and we can piece these together to give us a homotopy from ϕ^- and ϕ^+. This fixes setwise each α_V for $\alpha \subseteq \partial\Phi$, and can be modified to fix it pointwise. Combining with the homotopies from $\theta^\pm$ to $\phi^\pm$, we get a homology 3-chain in V, with boundary S, and whose image lies in a bounded neighbourhood of $T_V(\Phi,\eta)$. This image must include H. □

5. Topology of handlebodies

In this section, we make a few purely topological observations.

Let V be a topologically finite indecomposable 3-manifold (every embedded 2-sphere bounds a ball). Let Φ be a surface with non-empty boundary, $\partial\Phi$. Suppose we have an embedding, $\Phi \hookrightarrow V$, which is a homotopy equivalence. Then $\pi_1(V) \cong \pi_1(\Phi)$ and so V is homeomorphic to the interior of a handlebody [He].

Suppose that $\alpha \subseteq \partial\Phi$ is a boundary curve. By an *escaping homotopy* of α, we mean a proper map $\theta : \alpha \times [0,\infty) \longrightarrow V$ with $\theta(x,0) = x$ for all $x \in \alpha$, and with $\theta^{-1}(\partial\Phi) = \alpha \times \{0\}$. In other words, we homotope α away to infinity in the complement of $\partial\Phi$.

Let $H \subseteq V$ be a compact submanifold of V with $\Phi \subseteq \operatorname{int} H$, and with $V \setminus \operatorname{int} H \cong \partial H \times [0,\infty)$. We write $S = \partial H$. Note that H is a handlebody. It is easily seen that $S \hookrightarrow V \setminus \Phi$ is π_1-injective.

Let us suppose that each boundary curve admits an escaping homotopy that is injective and moreover that we can choose these homotopies to be pairwise disjoint and meeting Φ only in the corresponding boundary curve. If $\alpha \subseteq \partial\Phi$ is a boundary curve let $A_\alpha = \theta(\alpha \times [0,\infty))$. After isotopy, we can assume that A_α meets H in a compact annulus containing α. Let Φ_0 be the union of Φ together with each of these compact annuli. Then, Φ_0 is homeomorphic to Φ and is properly embedded in H, i.e. $\Phi_0 \cap \partial H = \partial\Phi_0$. Moreover, $\Phi_0 \hookrightarrow H$ is a homotopy equivalence. From this it follows that $\Phi_0 \hookrightarrow H$ extends to a homeomorphism of $\Phi_0 \times [-1,1]$ to H, with Φ_0 identified with $\Phi_0 \times \{0\}$. In particular, associated to each boundary curve, α, of Φ we have a curve $\alpha_S = S \cap A_\alpha$, homotopic to α in V. Note that $S = \Phi^- \cup \Phi^+$ with $\partial\Phi_0 = \Phi^- \cap \Phi^+$ equal to the union of these α_S, and with $\Phi^\pm$ homotopic to Φ in V. This expresses S as the double, $D\Phi$, of Φ. We also note that H is a regular neighbourhood of Φ in V.

We also note that if H' is any other submanifold of V of the type described above, then we can find a third, H'', with $H'' \supseteq H \cup H'$. Since their boundaries are all incompressible in $V \setminus \Phi$, it follows that $H'' \setminus \operatorname{int} H$ and $H'' \setminus \operatorname{int} H'$ are both products. It then follows that we can isotope H to H' in V, fixing $\partial\Phi$.

Lemma 5.1. *S is incompressible (i.e. π_1-injective) in $V \setminus \partial\Phi$.*

Proof. Suppose that the curve $\alpha \subseteq S$ is trivial in $V \setminus \partial\Phi$. Then α bounds a singular disc $D \longrightarrow V \setminus \partial\Phi$ which we can assume to be in general position with respect to Φ. Now, we can push this disc off Φ. To see this, consider any innermost component, β, of the preimage of Φ in D. This gives a curve in Φ which is trivial in V, hence in Φ. We can therefore push the subdisc of D bounded by β off Φ, eliminating β. After a finite number of such operations, we obtain a new disc bounding

α, disjoint from Φ. Since S is incompressible in $V \setminus \Phi$, it follows that α must be trivial in Φ. $\square$

We also note conversely that if $H \subseteq V$ is any compact submanifold with $\partial\Phi \subseteq \operatorname{int} H$ and $V \setminus \operatorname{int} H \cong S \times [0,\infty)$, then we can isotope Φ into H fixing $\partial\Phi$. After such an isotopy, H becomes a regular neighbourhood of Φ as above.

Let $\alpha \subseteq \partial\Phi$ be a boundary curve. Let $T(\alpha)$ be a regular neighbourhood of α in V. We can assume that $T(\alpha) \subseteq \operatorname{int} H$ and that $T(\alpha) \cap \Phi$ is a regular neighbourhood of α in Φ. Note that $\Phi \cap \partial T(\alpha)$ is a longitude of $\partial T(\alpha)$ (i.e. together with the meridian, it generates first homology of $\partial T(\alpha)$). In fact, its homotopy class is canonical, and so determines a framing of α, in the following sense.

Suppose that $\theta : \alpha \times [0,\infty) \longrightarrow V$ is an escaping homotopy. We can homotope θ in a small neighbourhood of $T(\alpha)$, fixing $\theta|\alpha$, so that $\theta^{-1}(T(\alpha))$ is a regular neighbourhood of $\alpha \times \{0\}$, which without loss of generality, we can take to be $\alpha \times [0,1]$, and such that $\theta|(\alpha \times [0,1])$ is injective. Let $\alpha_\theta = \theta(\alpha \times \{1\}) \subseteq \partial T(\alpha)$. Note that α_θ is also a longitude. We claim:

Lemma 5.2. *If θ and θ' are escaping homotopies (as above) then α_θ and $\alpha_{\theta'}$ are homotopic in $\partial T(\alpha)$.*

Proof. The homology class of $\alpha_\theta - \alpha_{\theta'}$ in $H_1(\partial T(\alpha);\mathbb{Z})$ is some multiple, n, of the meridian. We claim that $n = 0$. To see this, note that we can combine $\theta|(\alpha \times [1,\infty))$ and $\theta'|(\alpha \times [1,\infty))$ together with an annulus in $T(\alpha)$ to give a proper map of $\alpha \times \mathbb{R}$ into V. This defines a locally finite second homology class in V, which intersects the first homology class of α n times, and which is disjoint from $\partial\Phi \setminus \alpha$. But $\partial\Phi$ is trivial in $H_1(V;\mathbb{Z})$, and so $n = 0$. It follows that α_θ and $\alpha_{\theta'}$ are homologous hence homotopic in $\partial T(\alpha)$. $\square$

Note that (by hypothesis) there is an escaping homotopy of α that is injective and meets Φ only in α. In this case, α_θ and $\Phi \cap \partial T(\alpha)$ are disjoint, hence homotopic in $\partial T(\alpha)$. Thus, the class defined by Lemma 5.2 is the same as that of the preferred longitude defined earlier.

We can also use escaping homotopies to define a homotopy class of closed curves in S. More specifially, let $\theta : \alpha \times [0,\infty) \longrightarrow V$ be an escaping homotopy. We assume that θ is in general position with respect to S. Now $\theta^{-1}(H)$ is a compact neighbourhood of $\alpha \times \{0\}$ in $\alpha \times [0,\infty)$. In particular, there is an essential simple closed curve, $\beta \subseteq \alpha \times [0,\infty)$, with $\theta(\beta) \subseteq S$. Let α^S_θ be the homotopy class of $\theta(\beta)$ is S. (This implicitly assumes some choice of β.) Note that α^S_θ is homotopic to α in V.

Lemma 5.3. *Suppose that θ and θ' are escaping homotopies of α, and α^S_θ and $\alpha^S_{\theta'}$ are curves arising as above. Then α^S_θ and $\alpha^S_{\theta'}$ are homotopic in S.*

Proof. To each component, δ, of $\partial\Phi$, we associate a regular neighbourhood, $T(\delta)$, of δ as with α, above. We claim that $\partial T(\delta)$ is incompressible in $V \setminus \partial\Phi$. This can be seen explicitly as follows. Let $K = (\Phi \setminus \bigcup_\delta T(\delta)) \cup \bigcup_\delta \partial T(\delta)$. This is a 2-complex homeomorphic to a copy of Φ with a torus attached to each of the boundary components. Its inclusion into $V \setminus \partial\Phi$ is a homotopy equivalence. Indeed, each component of $V \setminus (K \cup \partial\Phi)$ is a product region. One can explicitly compute $\pi_1(V \setminus \partial\Phi) \cong \pi_1(K)$ as an amalgamated free product of $\pi_1(\Phi)$ with copies of $\mathbb{Z} \oplus \mathbb{Z}$ over $\mathbb{Z}$. In particular, the inclusion of each torus is π_1-injective.

Let W be the cover of $V \setminus \partial\Phi$ corresponding to the longitude of α in $\partial T(\alpha)$. There is a lift, C, of $\partial T(\alpha)$ which is a properly embedded bi-infinite cylinder. One can see explicitly from the above description that the inclusion $C \hookrightarrow W$ extends to a homeomorphism $C \times \mathbb{R} \longrightarrow W$ with C identified as $C \times \{0\}$. Moreover, there is a component, E, of the preimage of S which is another bi-infinite cylinder, whose inclusion into W is also a homotopy equivalence. (All other components of the preimage of S are discs.) The escaping homotopies, θ, θ', lift to maps of $\alpha \times [0, \infty)$ to W. These both cross C in a longitude by construction. We see they also cross E in essential curves which are lifts of α^S_θ and $\alpha^S_{\theta'}$. These are homotopic in W and hence in E. Projecting this homotopy back to S gives us a homotopy from α^S_θ to $\alpha^S_{\theta'}$ in S. □

Note that the homotopy class thus defined must be that of α_S described earlier. In particular, the curves α_S can be realised disjointly, and their union cuts S into two subsurfaces each homeomorphic to Φ.

In the above discussion, we took each $T(\alpha)$ to be a small regular neighbourhood of α. Suppose more generally that $T(\alpha)$ is any closed regular neighbourhood of α with $T(\alpha) \cap \partial\Phi = \alpha$. We can suppose that $T(\alpha)$ meets S in general position.

Lemma 5.4. *Any essential curve in $S \cap T(\alpha)$ is homotopic to α_S in S.*

Proof. Let γ be such a curve. Let $T \subseteq T(\alpha)$ be a smaller regular neighbourhood of α of the type described earlier. The longitude of ∂T is given by $\partial T \cap \Phi$. This is easily seen to be homotopic in $T(\alpha) \setminus \alpha$ to a curve in $\partial T(\alpha) \cap \Phi$. The latter is disjoint from γ. Thus γ is homotopic in $T(\alpha) \setminus \alpha$ to the longitude of ∂T.

The fact that γ is homotopic in S to α_S follows similarly as in the proof of Lemma 5.3, by passing to the cover, W of $V \setminus \partial\Phi$ corresponding to this longitude. $\square$

6. Completion of the proof in the closed surface case

The main result of this section will be Proposition 6.12. This justifies the hypotheses made in Proposition 4.1, thereby proving Theorem 2.1.

First we elaborate on some notions used earlier. The general principles behind these are well known.

Let M be a complete hyperbolic 3-manifold. Let η_0 be the 3-dimensional Margulis constant. Given any $\eta \in (0, \eta_0]$ we we write $T_M(\eta)$ for the η-thin part of M. We will refer to the components of $T_M(\eta)$ as "Margulis tubes", though for the moment, we can allow for cusps. This makes no essential difference to the argument.

Definition. By a *pleating surface* in M, we mean a 1-lipschitz map, $f : F \longrightarrow M$, where M is a hyperbolic surface (not necessarily compact) with ∂F totally geodesic in F.

Note that this maps the thin part of F (with respect to any positive constant) into the corresponding thin part of M. The "pleated surfaces" of Thurston are examples of maps of this sort. (See (C1) of Section 4.)

Lemma 6.1. *Suppose F is compact and f is π_1-injective. If $K \subseteq F$ is connected, and $f(K) \cap T_M(\eta) = \emptyset$, then the diameter of $f(K)$ in M is bounded in terms of η and $\xi(F)$.*

Proof. Note that $K \subseteq f^{-1}(T_M(\eta))$ lies in the η-thick part of F, so by Lemma 4.3, the diameter of K is bounded. It follows that the diameter of $f(K)$ is bounded. $\square$

One can generalise the above argument when f is not π_1-injective — we just need to know that there is a lower bound on the length of any closed geodesic in F that is trivial in M.

In the next lemma, we do not need to assume that f is π_1-injective.

Lemma 6.2. *Suppose F is compact. There is some $\eta_1 < \eta_0$ and an increasing map $\zeta : (0, \eta_1] \longrightarrow (0, \eta_0]$ depending only on $\xi(F)$ with the following property. Suppose that $f : F \longrightarrow M$ is a pleating surface and $x \in F$, with $f(x) \in T$, where T is an η-Margulis tube, with $\eta \leq \eta_1$. Then there is a simple closed curve $\gamma \subseteq F$ containing x, which is essential in F, and with $f(\gamma) \subseteq T'$, where $T' \supseteq T$ is the $\zeta(\eta)$-Margulis tube containing T. Moreover, the length of γ is bounded above in terms of η and $\xi(F)$.*

Proof. We can choose $\zeta(\eta)$ to that T' will always contain a large metric neighbourhood of T in M. Thus, $f^{-1}(T')$ contains a large metric ball around x in F. If this is large enough in relation to $\xi(F)$, then it cannot be a topological disc (otherwise, by Gauss-Bonnet, its area would exceed that of F). $\square$

We now return to our particular set up. Some analogous arguments can be found in Section 4 of [Mj1].

Let $M = \mathbb{H}^3/\Gamma$, where $\Gamma = \pi_1(\Sigma)$, $\Phi = \tilde{\Phi}/G \subseteq \Sigma$, where $G = \pi_1(\Phi) \leq \Gamma$, and $V = \mathbb{H}^3/G$, as in the previous sections. Let $\pi_M : \mathbb{H}^3 \longrightarrow M$, $\pi_V : \mathbb{H}^3 \longrightarrow V$ and $\pi_{VM} : V \longrightarrow M$ be the covering maps. Let $\eta \in (0, \eta_0]$. Note that G is closed under roots (i.e. $g \in \Gamma$ and $g^n \in G$ implies $n = 0$ or $g \in G$). From this it follows that if T_V is any η-Margulis tube in V, then $\pi_{VM}|T_V$ is injective, and $\pi_{VM}(T_V)$ is an η-Margulis tube in M. In fact:

Lemma 6.3. *Each η-Margulis tube in V has the form $T_V(\alpha, \eta)$, where V is homotopic to a simple closed curve in Φ.*

Proof. Let T_V be a tube in V. Then $\pi_{VM}(T_V)$ is a tube in M, and so has the form $T(\alpha, \eta)$ for some $\alpha \in X(\Sigma)$. But now α can be homotoped into Φ in Σ, and the homotopy lifts to V. $\square$

We write $T_V(\Phi, \eta)$ for the union of Margulis tubes in V. This is the η-thin part of V. We claim there are only finitely many such tubes. In other words:

Lemma 6.4. *$T_V(\Phi, \eta)$ is compact.*

Proof. Let α be any component of $\partial\Phi$. Then, $T_V(\alpha, \eta)$ is a component of $T_V(\Phi, \eta)$. Let $T_V(\delta, \eta)$ be any other component of $T_V(\Phi, \eta)$ (so that $\delta \in X_0(\Phi)$). We claim that $T_V(\delta, \eta)$ can be connected to $\partial T_V(\alpha, \eta)$ by a path, γ, in V, passing through boundedly many components of $T_V(\Phi, \eta)$ with length$(\gamma \backslash T_V(\Phi, \eta))$ bounded. The statement then follows by the local finiteness of the set of Margulis tubes (which holds in any hyperbolic manifold).

To prove the claim, note that (by the construction (C2) described in Section 4), we can find a pleating surface, $f : \Phi \longrightarrow V$, in the natural homotopy class, with $f(\alpha)$ and $f(\delta)$ closed geodesics. The diameter of each component of $f^{-1}(V \setminus T_V(\Phi, \eta))$ is bounded in terms of η and $\xi(\Phi)$ (Lemma 4.3). We can connect α to δ by a path passing through boundedly many such components. Its image in V gives us our path. $\square$

Let H be the convex core of V.

Lemma 6.5. *H is compact.*

Proof. Tameness [A, CalG] tells us that either V is geometrically finite, or simply degenerate. Since we are assuming there are no parabolics, in the former case, H is compact. We want to rule out the latter.

In the latter case, given any compact set, $K \subseteq V$, there is a pleating surface, $f : D\Phi \longrightarrow V$, such that $f(D\Phi)$ homologically separates K from the end of V. (This is essentially the definition of a simply degenerate end.) By Lemma 6.4, we can take $K \supseteq T_V(\Phi)$. In this case, the diameter of $f(D\Phi)$ is bounded above in terms of η and $\xi(\Phi)$. Taking the closure of the component of $V \setminus f(D\Phi)$ containing K, we get a compact subset, $W \subseteq V$, with the diameter of ∂W in V bounded above.

By taking a compact exhaustion, $(K_n)_n$, of V, we obtain in this way another compact exhaustion $(W_n)_n$, with the diameters of ∂W_n in V bounded above. Mapping down to M we get a compact exhaustion of M by sets $\pi_{VM} W_n$, again with the diameters of the boundaries bounded above. From this we deduce that M has one end. However, $M \cong \Sigma \times \mathbb{R}$, giving a contradiction. $\square$

It now follows that $S = \partial H$ is a hyperbolic surface, and $V \setminus \operatorname{int} H \cong S \times [0, \infty)$.

Remark. Lemma 6.5 can similarly be deduced using Canary's covering theorem [Can] as in [Mj1]. An immediate consequence is that any G-invariant subset of $\mathbb{H}^3$, in particular, $\tilde{T}(\tilde{\Phi}, \eta)$, is quasiconvex, though the constant may depend on the particular group.

Lemma 6.6. *There is a positive lower bound, depending only on η and g, on the length of any essential curve in S, which is trivial in V.*

Proof. Any disc bounding such a curve has to meet the cores of at least two Margulis tubes, or else intersect one such core essentially twice. Since any trivial curve bounds a disc of the same diameter, we easily get a lower bound on its length. $\square$

Let $\hat{\Phi}$ be the cover of Σ corresponding to Φ. We can identify $\Phi \subseteq \hat{\Phi}$ as a compact submanifold. Each $\alpha \subseteq \partial\Phi$ cuts off an annulus $A_\alpha \cong \alpha \times [0, \infty) \subseteq \hat{\Phi}$, with $\alpha \times \{0\}$ identified with α. Thus $\hat{\Phi} = \Phi \cup \bigcup_{\alpha \subseteq \partial\Phi} A_\alpha$.

We claim there is a proper embedding of $\hat{\Phi}$ in V which sends each curve, α, to the corresponding geodesic $\alpha_H \subseteq H$. To see this, recall that the set of Margulis tubes in M is unlinked [Ot]. In particular, we can find an embedding of $\hat{\Phi}$ in M which sends each α to α_M. We now lift this to V. We are then in the situation described in Section 5 (where the embeddings of A_α give us escaping homotopies). As observed there, after isotopy, we can assume that the embedding of Φ in V maps into H. (We are not making any claims here about the geometry of this embedding.)

Thus, to each $\alpha \subseteq \partial\Phi$, we have a preferred homotopy class α_S in $S = \partial H$, which we can assume to be realised as a closed geodesic. We aim to place an upper bound on the lengths of these α_S (Proposition 6.12). To this end, we focus on a particular component, α, of Φ. We will assume that η is small enough so that $\zeta^2(\eta) \leq \eta_0$.

First, we may suppose that $T(\alpha, \zeta(\eta)) \subseteq H$. For if not, $S \cap T(\alpha, \zeta(\eta)) \neq \emptyset$ and by Lemma 6.2, there is an essential simple closed curve, $\gamma \subseteq S$, with $\gamma \subseteq T(\alpha, \zeta^2(\eta))$, and with length$(\gamma)$ bounded above in terms of η and $\xi(\Phi) \leq \xi(\Sigma)$. By Lemma 5.4, γ is homotopic to α_S in S. This places a bound on α_S, so we are done.

Henceforth, we assume that $T(\alpha, \zeta(\eta)) \subseteq H$, so that $T(\alpha, \zeta(\eta)) \cap S = \emptyset$.

We now construct a specific map $f : \Sigma \longrightarrow M$ and a specific escaping homotopy of α as follows.

Let β be a multicurve in Σ, containing $\partial\Phi$, and maximal with the property that $l_M(\delta) \leq \eta$, for each component, δ, of β. Let $f : \Sigma \longrightarrow M$ be a pleating surface realising β (as in (C1) of Section 4), so that $f(\delta) = \delta_M$ for each δ. If $f(\Sigma) \cap T_M(\epsilon, \eta) \neq \emptyset$, we claim that $\epsilon \subseteq \beta$. To see this, note that $f^{-1}(T_M(\epsilon, \zeta(\eta)))$ contains an essential curve homotopic to ϵ. If ϵ is not a component of β, then $T_M(\epsilon, \zeta(\eta))$ is disjoint from each δ_M, and so this essential curve must be disjoint from β. It follows that ϵ cannot cross β in Σ, and so, $\beta \cup \epsilon$ is a multicurve. But $l_M(\epsilon) \leq \eta$, so by maximality of β, we have $\epsilon \subseteq \beta$ as claimed.

We lift f to a map $\hat{f} : \hat{\Phi} \longrightarrow V$. Note that each component of $\hat{f}^{-1}(T_V(\alpha, \zeta(\eta)))$ is either an annulus, possibly with a finite number of discs removed, parallel to α, or else homotopically trivial in $\hat{\Phi}$, not meeting $\hat{f}^{-1}(T_V(\alpha, \eta))$. We can therefore homotope $\hat{f}$ on the interior of $T_V(\alpha, \zeta(\eta))$ to obtain another map from $\hat{\Phi}$ to V which is bijective on the preimage of α. We let θ be the restriction of this map to A_α. Thus $\theta : A_\alpha \longrightarrow V$ is an escaping homotopy of V. It agrees with $\hat{f}$ outside $T_V(\alpha, \zeta(\eta))$.

The idea is to show that the intersection of $\theta(A_\alpha)$ with S contains a curve of bounded length, which will lie in the homotopy class α_S in S. Before bounding the length, we will bound the diameter.

Lemma 6.7. *$S \cap \theta(A_\alpha) \cap T_V(\Phi, \zeta(\eta)) = \emptyset$.*

Proof. Recall that θ agrees with $\hat{f}$ outside $T_V(\alpha,\zeta(\eta))$ and that $S\cap T_V(\alpha,\zeta(\eta))=\emptyset$. The statement is therefore equivalent to $S\cap\hat{f}(A_\alpha)\cap T_V(\Phi,\zeta(\eta))=\emptyset$. Again since $S\cap T_V(\alpha,\zeta(\eta))=\emptyset$, it is enough to show that $\hat{f}(A_\alpha)\cap T_V(\Phi,\zeta(\eta))\subseteq T_V(\alpha,\zeta(\eta))$. Suppose that $\theta(A_\alpha)\cap T_V(\beta,\zeta(\eta))\neq\emptyset$, for some curve, β, in Φ. Now $f(\Sigma)\cap T_M(\beta,\zeta(\eta))\neq\emptyset$, and so, by Lemma 6.2, $f^{-1}(T_M(\beta,\zeta^2(\eta)))$ contains a curve homotopic to β. Lifting to $\hat{\Phi}$, we see that $\hat{f}^{-1}(T_V(\beta,\zeta^2(\eta)))$ also contains a curve homotopic to β. This lies in A_α, and so β is homotopic to α. But, by assumption, $\theta(A_\alpha)\cap T_V(\alpha,\zeta(\eta))=\emptyset$, giving a contradiction. $\square$

Fix some $h>0$. We can asssume that the h-neighbourhood of any η-Margulis tube lies inside the corresponding $\zeta(\eta)$-tube.

Let B be the h-neighbourhood of $\theta^{-1}(H)$ in A_α. This is a compact set containing α. Now $B\setminus\theta^{-1}(H)$ has a connected component, C, which separates α from the end of A_α. We can write $\partial C=\partial_0 C\sqcup\partial_1 C$ where $\partial_0 C\subseteq\theta^{-1}(H)$, $d(\partial_0 C,\partial_1 C)=h$ and $C\subseteq N(\partial_0 C,h)$. Note that $\theta(\partial_0 C)\subseteq S$. Thus, by Lemma 6.7, we have:

Lemma 6.8. $\theta(\partial_0 C)\cap T_V(\Phi,\zeta(\eta))=\emptyset$.

Let $p:V\longrightarrow H$ be the nearest point projection. This is a 1-lipschitz retraction onto H.

Since p is 1-lipschitz and θ is 1-lipschitz outside $T_V(\alpha,\zeta(\eta))$, we see that $p(\theta(C))$ lies in a h-neighbourhood of $\theta(\partial_0 C)=p(\theta(\partial_0 C))$ in S. Since $\theta(C)\cap T_V(\Phi,\zeta(\eta))=\emptyset$, we have $p(\theta(C))\cap T_V(\Phi,\eta)=\emptyset$. Now C, hence $p(\theta(C))$ is connected. In summary, $p(\theta(C))$ does not meet the η-thin part of S and is connected. Also, in view of Lemma 6.6 we see that the diameter of $p(\theta(C))$ in S is bounded (see Lemma 6.1 and the subsequent remark). In particular, we deduce:

Lemma 6.9. *The diameter of $\theta(\partial_0 C)$ in S is bounded above in terms of η and g.*

We write $s_0=s_0(\eta,g)$ for this bound.

Let $\pi_\Sigma:\tilde{\Sigma}\longrightarrow\Sigma$ be the covering map. Recall that $\pi_{VM}:V\longrightarrow M$ is the covering map from V to M. We have $f:\Sigma\longrightarrow M$ and its lift $\hat{f}:\hat{\Phi}\longrightarrow V$. We see that $f\circ\pi_\Sigma=\pi_{VM}\circ\hat{f}$. Recall that $\theta:A_\alpha\longrightarrow V$ agrees with $\hat{f}$ on the exterior of $T_V(\alpha,\zeta(\eta))$, and so $f\circ\pi_\Sigma(\partial_0 C)=\pi_{VM}\circ\theta(\partial_0 C)$.

Lemma 6.10. *There is some $\eta'>0$, depending only on η and g, such that $\pi_\Sigma(\partial_0 C)$ lies in the η'-thick part of Σ.*

Proof. By Lemma 6.9, the diameter of $\theta(\partial_0 C)$ and hence that of $f\circ\pi_\Sigma(\partial_0 C)=\pi_{VM}\circ\theta(\partial_0 C)$ is at most s_0. Since it cannot lie entirely in an η-margulis tube, there is a bound on how deeply it can enter any such tube. Thus, it lies in the η'-thick part of M, where $\eta' > 0$ depends only on η and g. Since f is 1-lipschitz, it maps the η'-thin of Σ into that of M, and so the statement follows. □

Lemma 6.11. *The diameter of $\partial_0 C$ in A_α is bounded above in terms of η and g.*

Proof. First note that ∂A_α has length at most η, and so can be assumed to be isometrically embedded in $\hat{\Phi}$. It follows that A_α is also isometrically embedded in $\hat{\Phi}$. (That is, the induced metric on A_α is a path-metric.) Thus the statement is equivalent to showing that its diameter in $\hat{\Phi}$ is bounded.

Now, $\pi_\Sigma(\partial_0 C)$ is connected, and lies in the η'-thick part of Σ. Let $F \subseteq \Sigma$ be the component of the η'-thick part of Σ, containing $\pi_\Sigma(\partial_0 C)$. Now $f(F)$ cannot lie entirely in the η'-thin part of M and so we can choose some $x \in F$, with $f(x)$ in the η'-thick part. The diameter of F is bounded (by Lemma 4.3), so there is some s_1, depending only on η' and g, and hence only on η and g, such that $\pi_\Sigma(\partial_0 C)$ lies in an s_1-neighbourhood of x in Σ.

Let $Q \subseteq \hat{\Phi}$ be the preimage of x in $\hat{\Phi}$ under the covering map $\hat{\Phi} \longrightarrow \Sigma$. Let $Q_0 \subseteq Q$ be the set of point of Q a distance at most s_1 from $\partial_0 C$ in $\hat{\Phi}$. Thus $\partial_0 C$ lies in an s_1-neighbourhood of Q_0 in $\hat{\Phi}$.

Now the diameter of $\hat{f}(\partial_0 C) = \theta(\partial_0 C)$ is at most s_0 in S hence in V. Thus the diameter of $\hat{f}(Q_0)$ in V is at most $s_2 = s_0 + 2s_1$. Moreover, $\hat{f}|Q$ is injective. In fact, since Q maps into the η'-thick part of M, we see that if $y, z \in Q$ are distinct, $\hat{f}(y)$ and $\hat{f}(z)$ are distance at least η' apart in V. This therefore places an upper bound on the cardinality, $|Q_0| = |\hat{f}(Q_0)|$, depending only on η' and s_2, and hence only on η and g.

Let Λ be the graph with vertex set Q_0, where two points of Q_0 are deemed adjacent if they are distance at most $3s_1$ apart in $\hat{\Phi}$. Since $\partial_0 C$ is connected and the Hausdorff distance between $\partial_0 C$ and Q_0 is at most s_1, we see that Λ is connected. This places a bound on the combinatorial diameter of Λ, and hence on the diameter of $\partial_0 C$ in $\hat{\Phi}$. This in turn bounds the diameter of $\partial_0 C$ in A_α as claimed. □

Recall that C is connected, that it separates α from the end of A_α, and that $\partial C = \partial_0 C \sqcup \partial_1 C$ with $d(\partial_0 C, \partial_1 C) \geq h$. It now follows easily that there is a curve, $\gamma \subseteq C$ homotopic to α, with length $\gamma \leq l$, where l depends only on the bound of Lemma 6.11, hence only on η and g. Note that $\mathrm{length}(p(\theta(\gamma))) \leq l$. Now γ can be homotoped in C arbitrarily close to $\partial_0 C$. Recall that $\theta(C)$ lies outside H, so applying

$p \circ \theta$ to this homotopy, we also get a homotopy of $p(\theta(\gamma))$ in S to a curve arbitrarily close to $\theta(\partial_0 C)$. From the description in Section 5, we see that this homotopy class in S must be the same as α_S. Since α_S is geodesic, we get length$(\alpha_S) \leq l$.

This proves:

Proposition 6.12. *The length of each geodesic α_S in S is bounded above in terms of η and g.*

As discussed earlier, Proposition 6.12 now tells us that the constant, l featuring in the hypotheses of Proposition 4.1 can be taken to depend only on η and g. Thus, $H \subseteq N(T_V(\Phi, \eta), s)$, where s depends only on η and g. This proves Theorem 2.1 as discussed in Section 3.

7. The case with cusps

Let Σ, Γ, M and Φ etc. be as in the hypotheses of Theorem 2.2. Let $G \equiv \pi_1(\Phi) \leq \Gamma$ and $V = \mathbb{H}^3/G$. Let H be the convex core of V. We write $\Psi(V, \eta)$ for the η-non-cupsidal part of V. We abbreviate $\Psi(V) = \Psi(V, \eta_0)$ where η_0 is a fixed Margulis constant. We can assume that η_0 is chosen small enough so that ∂H meets each cusp in two horocyclic cusps of ∂H, which are totally geodesic in V. We outline the modifications to the previous sections needed to prove Theorem 2.2

No change is necessary in Section 3.

In Section 4, we partition $\partial\Phi = \partial_C\Phi \sqcup \partial_\Sigma\Phi$, where $\partial_\Sigma\Phi = \partial\Phi \setminus \Sigma$ and $\partial_C\Phi = \partial\Phi \cap \Sigma$. Then $\partial_C\Phi$ is in bijective correspondence with the set of cusps of V. We can assume that η_0 is chosen small enough so that ∂H meets each cusp in two horocyclic cusps of ∂H which are totally geodesic in V. Thus $H \cap \partial\Psi(V)$ consists of a disjoint union of annuli, each homotopic to a component of $\partial\Psi(M)$. The set $H \cap \Psi(V)$ will be a compact handlebody, whose complement in V is just a product. We are therefore in the topological situation described in Section 5, where $H \cap \Psi(V)$ now replaces H. We can write $\partial(H \cap \Psi(V)) = (\partial H \cap \Psi(V)) \cup (H \cap \partial\Psi(V)) = \Phi^+ \cup \Phi^-$, where $\Phi^+ \cap \Phi^- = (\partial\Phi)_S$ is a multicurve. We can assume that Φ is properly embedded in $H \cap \Psi(V)$, such that $\partial_C\Phi \subseteq \partial\Phi$ is a core of $H \cap \partial\Psi(V)$ (i.e. $H \cap \partial\Psi(V)$ retracts on the multicurve $\partial_C\Phi$).

Note also, that since we are assuming that Φ is a proper subsurface of Σ, $\partial_\Sigma\Phi \neq \emptyset$, and so we see that $\partial H \cap \Psi(V)$ is connected (being homeomorphic to the double of Φ along $\partial_\Sigma\Phi$). Now, $\Psi(V) \setminus \mathrm{int}(H)$ is just a product with boundary $\partial H \cap \Psi(V)$. From this, it follows that $\Psi(V)$ has only one end.

In the hypotheses of Proposition 4.1, it is only necessary to assume a bound, l, on the lengths of realisations of curves α in $\partial_\Sigma\Phi$. The conclusion remains unchanged. The proof is essentially the same. We need to modify the notion of a pleating surface in V. We use, more generally, a proper uniformly lipschitz map from $\Phi \setminus \partial_C \Phi$ into V, where $\Phi \setminus \partial_C \Phi$ carries a complete finite-area hyperbolic structure with totally geodesic boundary, $\partial_\Sigma \Phi$. Thus cusps of $\Phi \setminus \partial_C \Phi$ get sent to cusps of Φ.

No change is needed to the content of Section 5, though when we come to apply it in Section 6, the compact manifold, $H \cap \Psi(V)$ will take the place of the convex core (denoted H in Section 5).

In Section 6, we use pleating surfaces in the more general sense described above. Lemma 6.4 should now say that for any $\eta < \eta_0$ of $T_V(\Phi, \eta) \cap \Psi(V)$ is compact. This is equivalent to asserting that $T_V(\Phi, \eta)$ has finitely many components. For the proof, take α to be any component of $\partial_\Sigma \Phi$. Lemma 6.5 should now assert that $H \cap \Psi(V)$ is compact. For this, note that $\Psi(V)$ has only one end. If $H \cap \Psi(V)$ is not compact, then this end must be simply degenerate. We can then find a pleating surface, $D_\Sigma \Phi \longrightarrow V$, which avoids any given compact set, where $D_\Sigma \Phi$ is the surface obtained by doubling $\Phi \setminus \partial_C \Phi$ in its boundary, $\partial_\Sigma \Phi$. Projecting to M, this surface would have to avoid any other cusp of M (corresponding to any component of $\partial \Sigma \setminus \partial \Phi$), so we arrive at a similar contradiction. It then follows that ∂H is a finite-area hyperbolic surface, meeting $\partial \Psi(V)$ in disjoint horocycles. The last statement still holds if we replace $\Psi(V)$ by $\Psi(V, \eta)$ for any $\eta \leq \eta_0$. (We can also use Canary's covering theorem [Can].)

We let $\hat{\Phi}$ be the cover of $\Phi \setminus \partial_C \Phi$ in $\tilde{\Sigma}$ corresponding to Φ. Thus, we can write $\hat{\Phi} = (\Phi \setminus \partial \Sigma) \cup \bigcup_\alpha A_\alpha$, where α ranges over the components of $\partial_\Sigma \Phi$, and each A_α is a half-open annulus with α as its boundary component. We can now construct escaping homotopies of each α_V as before. We need from [Ot] the fact that the set of Margulis tubes in M are unlinked in $\Psi(M)$. The remainder of the argument proceeds similarly, where the constants now depend on g, p and η.

Finally, we remark that one can generalise Theorem 2.2 to allow for accidental cusps. In this case, a finite subset of $X(\Sigma)$ corresponds to a set of accidental $\mathbb{Z}$-cusps of M. We include these where appropriate in the definition of $T_M(\Phi, \eta)$. The conclusion still holds, and the proof is essentially the same. We just have the added technical complication that when we realise multicurves some of the components may degenerate to these accidental cusps. We can generalise the notion of a hyperbolic surface in M so that the domain has a “nodal structure”, that is, carries a complete finite are hyperbolic structure on the complement of these curves The curves themselves get sent to ideal points of the manifold (see Section 7 of [Bow3] for more details).

Again, we could also allow components of $\partial\Phi$ to correspond to cusps. This involves some further reinterpretation of earlier constructions. The ideas behind this are fairly standard, and we will not give details here.

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