Convergence groups and configuration spaces.

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0. Introduction.

In this paper, we give an account of convergence groups in a fairly general context, focusing on aspects for which there seems to be no detailed account in the literature. We thus develop the theory from a slightly different perspective to usual; in particular from the point of view of actions on spaces of triples. The main applications we have in mind are to (word) hyperbolic groups, and more generally to Gromov hyperbolic spaces. In later sections, we give special attention to group actions on continua.

The notion of a convergence group was introduced by Gehring and Martin [GeM1]. The idea is to axiomatise the essential dynamical properties of a Kleinian group acting on the ideal sphere of (real) hyperbolic space. The original paper thus refers directly only to actions on topological spheres, though most of the theory would seem to generalise to compact hausdorff spaces (or at least to compact metrisable spaces). The motivation for this generalisation stems from the fact that a (word) hyperbolic group (in the sense of Gromov [Gr]) acting on its boundary satisfies the convergence axioms. Indeed any group acting properly discontinuously on a complete locally compact (Gromov) hyperbolic space induces a convergence action on the ideal boundary of the space. More general accounts of convergence groups in this context can be found in [T2], [F1] and [F2].

There are essentially two equivalent definitions of "convergence group". The original, and most readily used, demands that every sequence of distinct group elements should have a "convergence subsequence" (or what we shall call a "collapsing subsequence") with very simple dynamics. The second definition, which we focus on here, demands that the induced action on the space of distinct triples should be properly discontinuous. This is, in many ways, a more natural formulation. The equivalence of these definitions for group actions on spheres is proved in [GeM2]. It would seem that their argument extends without change to actions on (metrisable) Peano continua. For the general case (compact hausdorff spaces) we shall need a slightly different approach. The second definition can be restricted to give us what we shall call "uniform convergence actions" — where the action on distinct triples is assumed also to be cocompact. The typical example of such an action is that of a hyperbolic group on its boundary. In fact, it is shown in [Bo6] that these are the only such examples. In some ways, this renders further study of uniform convergence actions superfluous. However certain properties of hyperbolic groups seem to fit most naturally into this dynamical context (see, for example [Bo2,Bo3,Bo5]), so it is appropriate to give, as far as possible, a purely dynamical treatment of some of these in the hope of finding broader applications (for example, to relatively hyperbolic groups). Indeed, introducing geometric considerations often does not seem to help significantly anyway.

There are various categories of spaces in which one might be interested. For example, with decreasing generality, we have compacta (compact hausdorff spaces), perfect compacta (compacta with no isolated points), continua (connected compacta) and Peano continua (locally connected continua). Every compactum contains a unique maximal perfect closed subset, which in all interesting cases is non-empty (i.e. for non-elementary groups). This is clearly preserved by any group action. We thus do not loose much by restricting to perfect compacta whenever this is convenient. Continua arise as cases of particular interest (for example, boundaries of one-ended hyperbolic groups). Peano continua are much easier to deal with. Most of the standard arguments concerning actions on spheres would seem to generalise unchanged to this context. The passage to the general case of compacta is sometimes slightly less trivial.

One of the main motives for writing this paper was to establish some of the groundwork for a deeper study of convergence actions on continua. These arise as important special cases of limit sets and ideal boundaries. There are a number of conjectures which assert that in certain circumstances, such continua are necessarily Peano continua — for example connected limit sets of geometrically finite Kleinian groups in any dimension (which one might generalise to relative hyperbolic groups) or finitely generated Kleinian groups in dimension 3. (The intersection of these cases, namely 3-dimensional geometrically finite Kleinian groups is already known, see [AnM].) It was also conjectured in [BesM] that the boundary of a one-ended hyperbolic group is locally connected. They showed that this is the case if there is no global cut point. The latter now follows from the results of [Bo2,Bo4,L,Sw]. A more general criterion for the non-existence of global cut points is given in [Bo5], which also has applications to geometrically finite Kleinian groups, and might shed some more light on the first conjecture mentioned above. With these, and other, potential applications in mind, it seems appropriate, as far as possible, to deal with general continua, without any local connectedness assumption.

There is another direction in which one might want to restrict the category of spaces under consideration. Most standard arguments (see [GeM1,T1,T2] etc.) make reference to convergence sequences, and thus effectively make some assumption about the order types of neighbourhood bases, or metrisability. Indeed, all likely applications are to metrisable spaces. However some natural constructions (for example those of Section 5) cannot be guaranteed to keep us in the metrisable category. For this reason, we shall avoid making any such hypothesis. Since we do not need any "diagonal sequence" arguments, this simply entails replacing the term "sequence" by "net", and "subsequence" by "subnet". The arguments can be translated back into more familiar terms simply by inverting this transformation.

As mentioned earlier, the original motivation for the study of convergence groups concerned Kleinian groups, i.e. groups acting properly discontinuously on hyperbolic space (see for example, [Mas] or [Ni]). More generally, one could consider groups acting on manifolds of pinched negative curvature. Many of the definitions concerning types of limit points etc. can be interpreted in the context of convergence groups. For example one can give a definition of geometrical finiteness intrinsic to the action of the group on its limit set (see [Bo1], generalising the description given in [BeaM].) Thus, a group is geometrically finite if and only if every limit point is a conical limit point or a "bounded"

parabolic fixed point. Such definitions make sense for convergence groups, though it's unclear to what extent the standard results generalise to this case, or indeed to what extent such generalisations would be genuinely useful. However, in certain cases, in particular uniform convergence actions (corresponding to convex cocompact Kleinian groups) such generalisations seem natural, and leads to some powerful techniques for their study.

As we have already mentioned, any uniform convergence group acting on a perfect compactum is hyperbolic, and the action is topologically conjugate to the action of the group on its boundary. One can, however, deduce many properties of such actions directly from these dynamical hypotheses, for example, the non-existence of parabolics. If the space is a Peano continuum, then we see that the group must be one-ended. One can go on to derive the JSJ splitting from a (mostly elementary) analysis of the local cut point structure [Bo3]. As mentioned earlier, the fact that any continuum admitting a uniform convergence action is necessarily locally connected requires a lot more work, including knowing the group is hyperbolic.

The specific cases of convergence groups acting on spheres have been much studied. In particular, the work of Tukia, Gabai and Casson and Jungreis [T1,Ga,CasJ] tells us that any convergence group acting on a circle is conjugate to a fuchsian group. (This result relies on the earlier analysis of convergence actions on the 2-disc by Martin and Tukia [MarT1]. It is, in turn, an essential step in the proof of the Seifert conjecture for 3-manifolds — see [Me].) One can similarly ask if every uniform convergence group Γ, acting on the 2-sphere, S^2, is conjugate to a cocompact Kleinian group. Some significant progress in this direction has been made by Cannon and coworkers (under the assumption that Γ is hyperbolic). See for example [CanS]. (We remark that it is known if Γ is quasiisometric to hyperbolic 3-space [CanC].) We also note that [Bo6] together with Stallings's theorem on ends [St] and Dunwoody's accessibility theorem, tells us that any uniform convergence group acting on a Cantor set is finitely generated virtually free.

We shall give precise definitions of convergence groups in Section 1. In fact, these definitions make sense for any set of homeomorphisms — closure under composition or inverses is irrelevant in this regard. We should note that we are using the term "convergence group" for what was called a "discrete convergence group" in [GeM1]. The term "discrete" has frequently been omitted in the subsequent literature, and the more general notion of "convergence group" described in the original paper will not concern us here.

An outline of the paper is as follows. In Section 1, we prove the equivalence of the two definitions of convergence group in a general setting. We show that a properly discontinuous action of a group on a complete locally compact hyperbolic space extends to a convergence action on the boundary. In Section 2, we give a brief outline of the standard results concerning convergence actions. In Section 3, we consider particular categories of limit points, in particular, conical limit points. In Section 4, we discuss various equivalent formulations of quasiconvexity for subgroups of a uniform convergence groups. In Section 5, we consider how properly discontinuous cocompact actions of a group can be compactified by uniform convergence actions. In Section 6, we consider connectedness properties of configuration spaces in continua. Of particular interest is the space of distinct triples. ¿From this, we can see directly that a group acting as a uniform convergence group on a Peano continuum is finitely generated and one-ended.

1. Convergence groups and spaces of triples.

In this section, we give two definitions of the "convergence property" of a set, Φ, of homeomorphisms of a compactum, M, to itself. In the case where Φ is a group, this defines the notion of a "convergence action" or "convergence group", though for most of this section, there will be no need to assume closure under composition or inverses.

The equivalence of these definitions is shown in [GeM2] for groups of homeomorphisms of spheres. Their argument would seem to generalise unchanged to the case where M is a (metrisable) Peano continuum. For the general case where M is any compactum, we shall use a slightly different argument.

We begin by introducing the space of distinct triples. Let M be (for the moment) any hausdorff topological space. We give the space of ordered triples, M^3, the product topology. Let $\Delta \subseteq M^3$ be the large diagonal, i.e. the (closed) subset of triples have at least two entries equal. Let $\Theta^0(M) = M^3 \setminus \Delta$ be the space of "distinct ordered triples".

There is a natural continuous surjective map $\Delta \longrightarrow M$ which sends any triple with at least two entries equal to x to the point $x \in M$. We denote the quotient by $\partial\Theta^0(M)$. Thus, $\partial\Theta^0(M)$ may be naturally identified with M. In fact, we can define an equivalence relation on M^3 by deeming two triples to be equivalent if two entries of the first triple are both equal to two entries of the second triple. Clearly, this relation is trivial (i.e. equality) on $\Theta^0(M)$. We may thus identify the quotient space as a union $\Theta^0(M) \cup \partial\Theta^0(M)$. This quotient is hausdorff, and contains $\partial\Theta^0(M)$ as a closed subset. If M is perfect, then $\Theta^0(M)$ is dense in $\Theta^0(M) \cup \partial\Theta^0(M)$. If M is compact, then so is $\Theta^0(M) \cup \partial\Theta^0(M)$. Also, if M is locally compact, then so is $\Theta^0(M)$.

Now, the symmetric group on three letters acts on M^3 by permuting the coordinates. This induces an action on $\Theta^0(M) \cup \partial\Theta^0(M)$, which is trivial on $\partial\Theta^0(M)$. We write the quotient as $\Theta(M) \cup \partial\Theta(M)$, where $\Theta(M)$ is the quotient of $\Theta^0(M)$. Again, $\partial\Theta(M)$ is closed, and may be naturally identified as M. We think of an element of $\Theta(M)$ as a "distinct (unordered) triple", i.e. a subset of M of cardinality 3.

Suppose now that M, N are compacta (compact hausdorff topological spaces), and that Φ is a set of homeomorphisms of M onto N. (Of course, we could take $M = N$, but we don't want to distinguish any preferred identity homeomorphism.) We write $\Phi^{-1} = \{\phi^{-1} \mid \phi \in \Phi\}$. Note that each $\phi \in \Phi$ induces a homeomorphism of $\Theta(M)$ onto $\Theta(N)$, which we shall also denote by ϕ.

In actual fact, we don't really want to assume that all the elements of Φ are distinct homeomorphisms (since we shall eventually want to allow for group actions with non-trivial kernel). To be more formal we should really view Φ as a collection of homeomorphisms with some indexing set, though to do so explicitly would only confuse our notation.

In what follows we shall assume that M (and hence N) has at least 3 points.

Definition : We say that Φ is *properly discontinuous on triples* if, for all compact subsets $K \subseteq \Theta(M)$ and $L \subseteq \Theta(N)$, the set $\{\phi \in \Phi \mid \phi K \cap L \neq \emptyset\}$ is finite.

Definition : If $\Phi' \subseteq \Phi$, $a \in M$ and $b \in N$, we say that Φ' is a *collapsing set* with respect to the pair (a, b) if, for all compact subsets $K \subseteq M \setminus \{a\}$ and $L \subseteq N \setminus \{b\}$, the set $\{\phi \in \Phi' \mid \phi K \cap L \neq \emptyset\}$ is finite.
We say that Φ' is a *collapsing set* if it is a collapsing set with respect to some pair (a, b).

Note that the pair (a, b) for a given collapsing set is uniquely determined. We shall refer to a and b, respectively, as the *repelling* and *attracting* points of the set Φ. (This terminology becomes more natural, when we reformulate this in terms of nets.) Note also that if Φ' is a collapsing set with respect to a pair (a, b), then $(\Phi')^{-1}$ is a collapsing set with respect to the pair (b, a).

Definition : We say that Φ has the *convergence property* if every infinite subset $\Phi' \subseteq \Phi$ contains a further infinite subset $\Phi'' \subseteq \Phi'$ which is a collapsing set.

We note that if Φ has the convergence property, then so does Φ^{-1}, as well as any infinite subset of Φ. This statement is also true of the property of being properly discontinuous on triples. Our first objective will be to show that these notions are equivalent:

Proposition 1.1 : *An infinite set of homeomorphisms of a compactum has the convergence property if and only if it is properly discontinuous on triples.*

The "only if" part is elementary and well-known. The "if" part is also known at least for spheres (and metrisable Peano continua). The general case involves a bit more work.

Before we give the proof we shall rephrase the definitions in a form that is more convenient to work with. The convergence property is usually phrased in terms of sequences. For the general case, we shall use nets. We begin by recalling a few standard (and not so standard) definitions concerning nets and subnets.

Let Z be any set. A *net* in Z is a map, $[n \mapsto z_n]$, from a directed set, $(D, \leq)$, to Z. We usually denote this by $(z_n)_n$, and don't bother to make explicit reference to the domain, D. If we use the same subscripts for two nets, then it's to be assumed that the domains are equal. A *final segment* of D is a subset of the form $\{n \in D \mid n \geq n_0\}$ for some $n_0 \in D$. We say that a property is true *for all sufficiently large n* if it is true for all n in some final segment. A subset of D is *cofinal* if it meets every final segment. We say that a net, $(z_n)_n$, is *wandering* if for all $z \in Z$, $z_n \neq z$ for all sufficiently large n.

If $(I, \leq)$ is another directed set, a map $[i \mapsto n(i)]$ from I into D is *cofinal* if for all $n \in D$, we have $n(i) \geq n$ for all sufficiently large i. A *subnet* of $(z_n)_n$ is a precomposition of $[z \mapsto z_n]$ with a cofinal map $[i \mapsto n(i)]$ from some directed set $(I, \leq)$. We write z_i for $z_{n(i)}$, and denote the subnet by $(z_i)_i$. Clearly every cofinal subset of D determines a subnet, but, in general, not every subnet need be of this form. In particular, a subnet

of a sequence need not be a subsequence. However, if $(z_n)_{n \in \mathbf{N}}$ is sequence, and $(z_i)_i$ is a subnet, we might abuse notation, and write z_{i+1} for $z_{n(i)+1}$ etc. Note that every subnet of a wandering net is wandering.

In what follows we shall assume standard results concerning nets, in particular, a space is compact if and only if every net has a convergent subnet (see, for example, [K]).

We have thus only changed language slightly from the usual. In the metrisable case, we can translate back into more familiar terms by replacing "net" by "sequence" and "subnet" by "subsequence". A "wandering net" can be translated as a "sequence of distinct elements".

We can now reformulate the definitions given above. To say that Φ is properly discontinuous on triples is equivalent to the following hypothesis. Suppose that $(\phi_n)_n$ is a wandering net of elements of Φ, and that $(x_n)_n$, $(y_n)_n$ and $(z_n)_n$ are nets in M. Suppose that $x_n \to x$, $y_n \to y$, $z_n \to z$, $\phi_n x_n \to x'$, $\phi_n y_n \to y'$ and $\phi_n z_n \to z'$. If x, y, z are all distinct, then x', y', z' cannot all be distinct (and so also conversely).

To reformulate the convergence property, we proceed as follows. We say that a net, $(\phi_i)_i$, of elements of Φ is a *collapsing net* if there are points $a \in M$ and $b \in N$ such that the net of maps $\phi_i | M \setminus \{a\}$ converges locally uniformly to the point b. We shall denote this by $\phi_i | M \setminus \{a\} \twoheadrightarrow b$. Note that a collapsing net is necessarily wandering. Local uniform convergence can, in turn, be expressed in terms of nets. Thus, $\phi_i | M \setminus \{a\}$ does not converge locally uniformly to b if and only if there is some subnet $(\phi_j)_j$ of $(\phi_i)_i$, and a net $(x_j)_j$ of points of M such that $x_j \to x$ and $\phi_j x_j \to x'$, where $x \in M \setminus \{a\}$ and $x' \in N \setminus \{b\}$.

We see that Φ has the convergence property if and only if every wandering net of elements of Φ has a collapsing subnet.

In what follows, we shall freely pass to subnets without necessarily changing notation. We justify such liberties with phrases such as "without loss of generality".

We now set about proving Proposition 1.1. One direction is easy:

Lemma 1.2 : *If Φ has the convergence property, then it is properly discontinuous on triples.*

Proof : Suppose $(\phi_n)_n$ is a wandering net of elements of Φ. Suppose we can find nets $(x_n)_n$, $(y_n)_n$ and $(z_n)_n$, of elements of M such that $x_n \to x$, $y_n \to y$, $z_n \to z$, $\phi_n x_n \to x'$, $\phi_n y_n \to y'$ and $\phi_n z_n \to z'$, where $x, y, z \in M$ are all distinct, and $x', y', z' \in N$.

After passing to a collapsing subnet, we can find points $a \in M$ and $b \in N$ such that $\phi_n | M \setminus \{a\} \twoheadrightarrow b$. Moreover, we can assume that $x, y \neq a$. It follows that $\phi_n x_n \to b$ and $\phi_n y_n \to b$. Thus, $x' = y' = b$. $\diamondsuit$

For the purposes of proving the converse, we shall introduce the following notation. If $a, b \in N$, and $(u_n)_n$ is a net in N, we shall write $u_n \to \{a, b\}$ to mean that for all neighbourhoods $O \ni a$ and $U \ni b$, we have $u_n \in O \cup U$ for all sufficiently large n.

Let's now suppose that Φ is properly discontinuous on triples. In the following lemmas, $(x_n)_n$, $(y_n)_n$, $(z_n)_n$ and $(w_n)_n$ are assumed to be nets in M, and $(\phi_n)_n$ is a wandering net in Φ.

Lemma 1.3 : *Suppose $x_n \to x$, $y_n \to y$, $z_n \to z$, with x, y, z distinct. Suppose $\phi_n x_n \to x'$ and $\phi_n y_n \to y'$ with $x' \neq y'$. Then $\phi_n z_n \to \{x', y'\}$.*

Proof : Otherwise some subnet of $\phi_n z_n$ would converge to a point $z' \notin \{x', y'\}$. $\diamond$

Lemma 1.4 : *Suppose $x_n \to x$, $y_n \to y$, $z_n \to z$, with x, y, z distinct, and that $\phi_n x_n \to a$, $\phi_n y_n \to a$ and $\phi_n z_n \to b \neq a$. If $w_n \to w \neq z$, then $\phi_n w_n \to \{a, b\}$.*

Proof : Without loss of generality, $w \neq y$, so we can apply Lemma 1.3 (replacing x_n by z_n, and z_n by w_n). $\diamond$

Lemma 1.5 : *Suppose $x_n \to x$, $y_n \to y$, $z_n \to z$ and $w_n \to w$, with x, y, z, w all distinct. Suppose $\phi_n x_n \to a$, $\phi_n y_n \to a$, $\phi_n z_n \to b$ and $\phi_n w_n \to b$. Then $a = b$.*

Proof : Choose any $c \in N \setminus \{a, b\}$, and let $u_n = \phi_n^{-1} c$. Passing to a subnet, $(u_n)_n$ can be assumed to converge to some point $u \in M$. Now, either $u \notin \{x, y\}$ or $u \notin \{z, w\}$. If $a \neq b$, then applying Lemma 1.4, we derive, either way, the contradiction that $c = \phi_n u_n \to \{a, b\}$. $\diamond$

Lemma 1.6 : *Suppose $x, y, z \in M$ are distinct, and $z_n \to z$. Suppose that $\phi_n x \to a$, $\phi_n y \to a$ and $\phi_n z_n \to b \neq a$. Then $\phi_n | M \setminus \{z\}$ converges locally uniformly to a.*

Proof : First, we prove pointwise convergence. Suppose $w \in M \setminus \{z\}$. By Lemma 1.4, we have $\phi_n w \to \{a, b\}$. If $\phi_n w \not\to a$, then, passing to a subnet, we can suppose that $\phi_n w \to b$, contradicting Lemma 1.5.

To prove locally uniform convergence, suppose (maybe after passing to a subnet) that $w_n \to w \neq z$. By pointwise convergence, we can suppose that $w \notin \{x, y\}$. (Otherwise replace x or y by some other point of $M \setminus \{z\}$.) Again by Lemma 1.4, we have $\phi_n w_n \to \{a, b\}$. If $w_n \not\to a$, we get a contradiction to Lemma 1.5 as before. $\diamond$

Lemma 1.7 : Φ *has the convergence property.*

Proof : Let $(\phi_n)_n$ be any wandering net in Φ. We want to find a collapsing subnet.

Choose any triple x, y, z of distinct elements of M. Passing to a subnet, and permuting x, y, z if necessary, we can assume that $\phi_n x \to a$, $\phi_n y \to a$ and $\phi_n z_n \to b$ for some $a, b \in N$.

If $a \neq b$, then Lemma 1.6 tells us immediately that $\phi_n | M \setminus \{z\} \twoheadrightarrow a$.

We can thus assume that $a = b$. Choose any point $c \in N \setminus \{a\}$, and let $w_n = \phi_n^{-1} c$. Passing to a further subnet, we can suppose that $w_n \to w \in M$. Without loss of generality, $w \notin \{x, y\}$. In this case, Lemma 1.6 tells us that $\phi_n | M \setminus \{w\} \twoheadrightarrow a$. $\diamond$

Lemmas 1.2 and 1.7 together prove Proposition 1.1. In fact, we can strengthen Lemma 1.7 as follows:

Proposition 1.8 : *If Φ is properly discontinuous on triples, then Φ has the convergence property on $\Theta(M) \cup \partial\Theta(M)$.*

Proof : We know, by Lemma 1.7, that any wandering net in Φ has a collapsing subnet in $M \equiv \partial\Theta(M)$. We are thus reduced to considering a net, $(\phi_n)_n$, in Φ such that $\phi_n|\partial\Theta(M)\setminus\{a\} \twoheadrightarrow b$, for some $a \in \partial\Theta(M)$ and $b \in \partial\Theta(N)$. We claim that $\phi_n|(\Theta(M)\cup\partial\Theta(M))\setminus\{a\} \twoheadrightarrow b$.

Suppose that $(\phi_i)_i$ is any subnet, and $(\theta_i)_i$ is a net in $\Theta(M)\cup\partial\Theta(M)$, which converges to some $\theta \in \Theta(M)\cup\partial\Theta(M)\setminus\{a\}$. We claim that $\phi_i\theta_i$ converges to b. We can partition the domain of the net into two subdomains depending on whether θ_i lies in $\Theta(M)$ or $\partial\Theta(M)$. This gives us (at most) two subnets, and it's enough to verify the claim for each of these. (Of course there's no reason to suppose that both subsets of the domain are cofinal, but if one isn't then there's nothing to verify in that case.) In fact, we know by construction that the claim is true for the subnet lying in $\partial\Theta(M)$, so we can assume, without loss of generality that $\theta_i \in \Theta(M)$ for all i. We write $\theta_i = \{x_i, y_i, z_i\}$.

Suppose first, that $\theta \in \Theta(M)$. Write $\theta = \{x, y, z\}$. We can assume that $x_i \to x$, $y_i \to y$ and $z_i \to z$ (since we are free to label the enties in the triple θ_i as we choose). Also without loss of generality, $x, y \neq a$. It follows that $\phi_i x_i \to b$ and $\phi_i y_i \to b$, and so $\phi_i\theta_i \to b$ in $\Theta(M)\cup\partial\Theta(M)$ as claimed.

We can thus assume that $\theta \in \partial\Theta(M)\setminus\{a\}$, so that θ corresponds to some point $x \in M\setminus\{a\}$. We can assume that $x_i \to x$ and $y_i \to x$. Since $x \neq a$, we have $\phi_i x_i \to a$ and $\phi_i y_i \to a$ in M. Thus, again $\phi_i\theta_i \to b$ as claimed. $\diamondsuit$

This concludes the basic observations about sets of homeomorphisms. The cases of interest here concern group actions.

Suppose that M is a compactum, and that Γ is a group acting by homeomorphism on M.

Definition : We say that Γ is a *convergence group* (or that the action is a *convergence action*) if, as a set of homeomorphisms, it has the convergence property.

We see that Γ is a convergence group if and only if the induced action on $\Theta(M)$ is properly discontinuous. Moreover, this implies that the induced action on $\Theta(M)\cup\partial\Theta(M)$ is also a convergence action.

There are two subtleties we should remark upon. The first is that we have not assumed that Γ acts effectively. Thus, we should more formally view the set of homeomorphisms in the above definitions as a collection indexed by Γ. In any case the definitions imply that the action should have finite kernel, so the distinction is not really important. The second point is that we have assumed that M has at least 3 elements. The appropriate definition of a convergence action on a smaller set may be open to debate, but it would seem natural to allow any action on a singleton, and any virtually cyclic action on a pair.

The following is a trivial, but useful observation:

Lemma 1.9 : *Suppose the group Γ acts by homeomorphism on locally compact hausdorff spaces X and Y. Suppose $f : Y \longrightarrow X$ is a proper surjective Γ-equivariant map. Then Γ acts properly discontinuously on X if and only if it acts properly discontinuously on Y. Also, Γ acts cocompactly on X if and only if it acts cocompactly on Y.* $\diamond$

We finish this section with two applications of this. The first concerns quotient spaces.

Suppose M, N are compacta, and $f : M \longrightarrow N$ is surjective. Let $\Theta_N(M) \subseteq \Theta(M)$ be the subset of triples $\{x, y, z\}$ such that fx, fy, fz are all distinct. We see that f induces a natural surjective map, $\Theta f : \Theta_N(M) \longrightarrow \Theta(N)$, given by $\Theta f(\{x, y, z\}) = \{fx, fy, fz\}$.

Suppose now that Γ acts on M and N, and that f is Γ-equivariant. It follows that $\Theta_N(M)$ is a Γ-invariant subset of $\Theta(M)$, and that Θf is Γ-equivariant. If Γ acts properly discontinuously on $\Theta(M)$, then it does so on $\Theta_N(M)$, and hence, by Lemma 1.9, on $\Theta(N)$. We deduce:

Proposition 1.10 : *Suppose that Γ acts on the compacta M and N, and that $f : M \longrightarrow N$ is a Γ-equivariant map. If Γ acts as a convergence group on M, then it acts as a convergence group on N.* $\diamond$

One can can give an alternative (perhaps simpler) proof of this result using the collapsing subsequence definition instead. (This is set out explicitly in [Bo2].)

The second application involves induced action on boundaries of hyperbolic spaces, as defined by Gromov [Gr]. For the necessary background, see, for example, [GhH].

Suppose that (X, d) is a complete, locally compact path-metric space which is (Gromov) hyperbolic. Thus, any closed metric ball in X is compact. Also, X can be compactified in a natural way by adjoining its (Gromov) boundary, ∂X. Thus, $X \cup \partial X$ carries a natural compact topology. It is also metrisable, though does not admit any preferred metric.

Suppose a group Γ acts properly discontinuously and isometrically on X. We get an induced action by homeomorphism on $X \cup \partial X$. We claim that this is a convergence action. For the purposes of future reference, we split this into two parts.

Lemma 1.11 : *Γ acts as a convergence group on ∂X.*

Proof : Let k be the constant of hyperbolicity of X (in the sense that for any geodesic triangle in X, there is a point a distance at most k from each of its edges). Let $Y \subseteq X \times \Theta(\partial X)$ be the subset of pairs, $(a, \{x_1, x_2, x_3\})$ such that there exist biinfinite geodesics, $\alpha_1, \alpha_2, \alpha_3$, with α_i connecting x_i to x_{i+1} (with subscripts mod 3) such that $d(a, \alpha_i) \leq k$ for each $i \in \{1, 2, 3\}$. Thus, Y is a closed subset of $X \times \Theta(\partial X)$. Moreover, the natural projections of Y to X and to $\Theta(\partial X)$ are both proper and surjective. Also the whole construction is natural and hence Γ-equivariant. Since Γ acts properly discontinuously on X, it follows, by Lemma 1.9, that it does so also on Y and hence on $\Theta(\partial X)$. $\diamond$

Proposition 1.12 : *Γ acts as a convergence group on $X \cup \partial X$.*

Proof : (Since $X \cup \partial X$ is metrisable, we may as well phrase everything in terms of sequences.)

Suppose that $(\gamma_n)_n$ is a sequence of distinct elements of Γ. Since Γ acts as a convergence group on ∂X (Lemma 1.11), we can find a subsequence, $(\gamma_i)_i$ and $a, b \in \partial X$, such that $\gamma_i | \partial X \setminus \{a\} \twoheadrightarrow b$. We claim that $\gamma_i | (X \cup \partial X) \setminus \{a\} \twoheadrightarrow b$.

To see this, suppose that $K \subseteq (X \cup \partial X) \setminus \{a\}$ is compact. Suppose that $(x_i)_{i \in \mathbf{N}}$ is any sequence in K. We can find a compact subset $L \subseteq \partial X \setminus \{a\}$ and points $y_i, z_i \in L$ such that each x_i lies in the (closed) biinfinite geodesic joining y_i to z_i. Now $y_i \to b$ and $z_i \to b$, and so it follows easily that $x_i \to b$ as required. $\diamond$

This result can be compared with Proposition 5.6, where Γ is assumed to act cocompactly, but X is not assumed to be metrisable.

We finish this section by introducing "uniform convergence groups" to which we shall return again later. Suppose that Γ acts by homeomorphism on a perfect compactum, M.

Definition : We say that Γ is a *uniform convergence group* if it acts properly discontinuously and cocompactly on the space of distinct triples, $\Theta(M)$.

We refer to the action as a "uniform convergence action". The typical examples arise as boundaries of hyperbolic groups, as we shall see below.

Suppose that Γ is a (word) hyperbolic group. Its boundary, $\partial \Gamma$, is a compact metrisable space, on which Γ acts by homeomorphism. Indeed, if (X, d) is any complete locally compact hyperbolic space, and Γ acts properly discontinuously and cocompactly on X, then we may naturally identify the (Gromov) boundary, ∂X with $\partial \Gamma$. The typical example of such an X is the Cayley graph of the given hyperbolic group with respect to any finite generating set. (One could also take the Rips complex etc., see [GhH,BesM].)

We already know (Lemma 1.11) that Γ acts as a convergence group on $\partial \Gamma$. This is also shown in and [F1] and [T2]. In this case our previous argument gives us the additional information:

Proposition 1.13 : *A hyperbolic group Γ acts as a uniform convergence group on its boundary, $\partial \Gamma$.*

Proof : Let X be a Cayley graph of Γ, so that we can equivariantly identify $\partial \Gamma$ and ∂X. As in the proof of Lemma 1.11, we construct a locally compact hausdorff space, Y, and proper equivariant surjections of Y to X and to $\Theta(\partial X)$. Since the action on X is properly discontinuous and cocompact, we see, by Lemma 1.9, the same is true of the action on $\Theta(\partial X)$. $\diamond$

This result well known (though I've not found an explicit reference). The converse was, for a time, an open problem, though it appears that Gromov had long been confident that this was indeed true. A proof is given in [Bo6].

2. General properties of convergence groups.

In this section, we briefly outline how one may develop the theory of convergence groups on general compacta. Most of the results stated here are well known, and accounts can be found in [GeM1] and [T2] (see also [T1], [F1] and [F2]). Some of these are given in slightly restricted contexts, though the arguments would seem to generalise unchanged. The proofs are typically based on the "collapsing net" (or "convergence subsequence") definition of convergence group.

Typically, the development proceeds via a classification of elements according to their dynamics, a discussion of "elementary" subgroups, the partition of the space into limit set and discontinuity domain etc. Here, we shall only concern ourselves with aspects relevant to the rest of this paper.

Suppose that Γ acts as a convergence group on the compactum, M, with $\operatorname{card} M \geq 3$. Given $\gamma \in \Gamma$, we write $\operatorname{fix} \gamma = \{x \in M \mid \gamma x = x\}$.

Definition : We say that an element of Γ is *elliptic* if it has finite order.

We say that $\gamma \in \Gamma$ is *parabolic* if it has infinite order, and $\operatorname{card} \operatorname{fix} \gamma = 1$.

We say that $\gamma \in \Gamma$ is *loxodromic* if it has infinite order, and $\operatorname{card} \operatorname{fix} \gamma = 2$.

Clearly these possibilities are mutually exclusive. The following is the most basic result about convergence groups. The proof we give here is more or less copied from [T2].

Lemma 2.1 : *Every element of Γ is elliptic, parabolic or loxodromic.*

Proof : Suppose $\gamma \in \Gamma$ has infinite order. Since $\langle \gamma \rangle$ acts properly discontinuously on distinct triples, we see that $\operatorname{card} \operatorname{fix} \gamma \leq 2$. It thus suffices to show that $\operatorname{fix} \gamma \neq \emptyset$.

Consider the sequence of elements $(\gamma^n)_{n \in \mathbf{N}}$. There is a subnet, $(\gamma^i)_i$, and points $a, b \in M$ such that $\gamma^i | M \setminus \{a\} \twoheadrightarrow b$. Choose any $c \in M \setminus \{a, \gamma^{-1}a\}$. Now, $\gamma^i c \to b$, so $\gamma^{i+1}c = \gamma\gamma^i c \to \gamma b$. But, $\gamma^{i+1}c = \gamma^i\gamma c \to b$, since $\gamma c \neq a$. Thus, $\gamma b = b$. $\diamond$

One can go on to show that, in fact, $(\gamma^n)_{n \in \mathbf{N}}$ is itself a collapsing sequence. In particular, $\langle \gamma \rangle$ acts properly discontinuously on $M \setminus \operatorname{fix} \gamma$.

If γ is parabolic with fixed point p, we see that, for all $x \in M$, $\gamma^n x \to p$ as $n \to \infty$ and as $n \to -\infty$. If γ is loxodromic, we can write $\operatorname{fix} \gamma = \{\operatorname{fix}^+ \gamma, \operatorname{fix}^- \gamma\}$, such that $\gamma^n | M \setminus \{\operatorname{fix}^- \gamma\} \twoheadrightarrow \operatorname{fix}^+ \gamma$. It's not hard to see that, in this case, $\langle \gamma \rangle$ acts cocompactly on $M \setminus \operatorname{fix} \gamma$. Note that every power of a parabolic is parabolic, and every power of a loxodromic is loxodromic.

It's known that a loxodromic cannot share a fixed point with a parabolic. Also if two loxodromics share a fixed point, then they have both fixed points in common. Moreover the setwise stabiliser of any pair of points is virtually cyclic. (For proofs, see for example [T2].) We may summarise these results as follows:

Lemma 2.2 : *Suppose an infinite subgroup, $G \leq \Gamma$ fixes some point $p \in M$. Then, G either consists entirely elliptics and parabolics, or consists entirely of elliptics and loxodromics. In the latter case G also fixes some other point, $q \in M \setminus \{p\}$, and is virtually cyclic.* ◇

These cases are mutually exclusive. We refer to them respectively as "parabolic" and "loxodromic". In fact, in the parabolic case, G acts properly discontinuously on $M \setminus \{p\}$. In the loxodromic case, it acts properly discontinuously and cocompactly on $M \setminus \{p, q\}$.

We shall refer to a subgroup $G \leq \Gamma$, as *elementary* if it is finite, or preserves setwise a nonempty subset of M with at most 2 elements. It is shown in [T2] that every non-elementary subgroup contains a free subgroup of rank 2.

It is conceivable in the "parabolic" case of Lemma 2.2, that G may contain only elliptic elements. There are no other possibilities for infinite torsion subgroups of Γ.

We remark that, from the result of [D2], any finitely generated inaccessible group must contain an infinite torsion subgroup. If we can rule out such possibilities (for example, for convergence actions on the 2-sphere), we can deduce that Γ is accessible.

Note that any element which commutes with a loxodromic must preserve setwise its fixed point set. ¿From this, it's a fairly easy deduction that:

Proposition 2.3 : *Any infinite virtually abelian subgroup of Γ has a subgroup of index at most 2 which fixes a point.* ◇

The following is shown in [T2]:

Lemma 2.4 : *Suppose that $U \subseteq M$ is open, with closure, $\bar{U}$. Suppose $\gamma \in \Gamma$ with $\gamma \bar{U} \subseteq U$. Then γ is loxodromic (with $\mathrm{fix}^+ \gamma \in U$ and $\mathrm{fix}^- \gamma \in M \setminus \bar{U}$).* ◇

The idea of the proof is to note that the sets $\bigcap_{n=0}^{\infty} \gamma^n \bar{U}$ and $\bigcap_{n=0}^{\infty} \gamma^{-n}(M \setminus U)$ are non-empty disjoint closed γ-invariant subsets. It's then easy to see that they must both be singletons, and hence fixed points of γ.

In particular, we deduce:

Lemma 2.5 : *If $(\gamma_n)_n$ is a net in Γ with $\gamma_n | M \setminus \{a\} \twoheadrightarrow b$, where $a \neq b$. Then, γ_n is loxodromic for all sufficiently large n.*

Proof : Let U be an open neighbourhood of b with $a \notin \bar{U}$. For all sufficiently large n, we have $\gamma_n \bar{U} \subseteq U$. For such n, γ_n is loxodromic. ◇

The next natural step in working with convergence groups is to define a natural partition of M into a limit set, Λ, and discontinuity domain, $\Omega = M \setminus \Lambda$. The *limit set* can be defined as the set of limit points, where a *limit point* is an accumulation point of a Γ-orbit. In other words, $x \in \Lambda$ if and only if there is a net, $(\gamma_n)_n$, in Γ, and a point $y \in M \setminus \{x\}$, such that $\gamma_n y \to x$. Thus, Λ is closed, and Ω is open. If Γ is infinite, then Λ is non-empty. If we assume that Γ is non-elementary, then Λ is perfect and Γ acts minimally on Λ. In fact, Λ is the unique minimal non-empty closed Γ-invariant subset of M. In contrast, Γ acts properly discontinuously on Ω. (Many actions we will be considering will

be non-elementary and minimal, i.e. $\Omega = \emptyset$. Note that this implies that M is perfect.) In the next section we shall be considering particular classes of limit points.

3. Conical limit points.

In studying Kleinian groups, it has proved important to distinguish different classes of limit points. These are discussed, for example, in [Mas] and [Ni]. Obvious examples of such classes are parabolic and loxodromic fixed points. A particularly important class (including the loxodromic fixed points) are "conical limit points" (also known as "radial limit points" or "points of approximation"). For example, they arise naturally in the study of conformal densities (as discussed in [Ni]). Also, one can characterise the property of geometrical finiteness dynamically, by demanding that every limit point is a conical limit point or a "bounded" parabolic fixed point (see [BeaM,Bo1]). In the context of convergence groups acting on spheres, conical limit points have appeared in [MarT1] and [MarT2].

In this section, we define a natural notion of conical limit point for convergence groups, which reduces to the standard notion in the case of Kleinian groups. It will be immediate from the definition that, in the case of a uniform convergence action, every point is a conical limit point. For Kleinian groups, the converse also holds. (This amounts to the statement that a geometrically finite group with no parabolics and empty discontinuity domain is cocompact.) I don't know if this converse is true in general.

Suppose, then, that Γ acts as a non-elementary convergence group on a compactum, M.

Definition : A point $x \in M$ is a *conical limit point* if there are nets $(x_n)_n$ and $(\gamma_n)_n$ in $M \setminus \{x, y\}$ and Γ respectively with $x_n \to x$ such that there exists $y \in M \setminus \{x\}$ such that $\gamma_n(x, y, x_n)$ remains in a compact subset of $\Theta^0(M)$.

(In fact, as we shall see, we get an equivalent definition if we replace the phrase "there exists $y \in M \setminus \{x\}$" by "for all $y \in M \setminus \{x\}$".)

Note that we can assume that $\gamma_n(x, y, x_n)$ converges on some point, $(a, b, c) \in \Theta^0(M)$, and that $(\gamma_n)_n$ is a collapsing net. In fact, we must have $\gamma_n|M \setminus \{x\} \twoheadrightarrow b$. (To see this, suppose $\gamma_n|M \setminus \{x'\} \twoheadrightarrow b'$, so that $\gamma_n^{-1}|M \setminus \{b'\} \twoheadrightarrow x'$. Now $(\gamma_n x_n)_n$ and $(\gamma_n x)_n$ converge on different points, at least one of which must lie in $M \setminus \{b'\}$, whereas, their images under γ_n^{-1} both converge on x. It follows that $x = x'$. Now, $\gamma_n y \to b$, but $\gamma_n^{-1}(\gamma_n y) \not\to x$, so we must have $b = b'$.) Note that if $z \in M \setminus \{x\}$, then $\gamma_n(x, z, x_n) \to (a, b, c)$. This justifies our earlier remark about quantifiers. Note also that x is an accumulation point of some Γ-orbit. We see:

Proposition 3.1 : *A conical limit point is a limit point.* $\diamondsuit$

We should note that the property of being a conical limit point is intrinsic to the action of Γ on the limit set, Λ. (That is, a point $x \in \Lambda$ is conical limit point for the action of Γ on M if and only of it is a conical limit point for the action of Γ restricted to Λ.) This is easy to see, noting that $\Theta^0(\Lambda)$ is a closed subset of $\Theta^0(M)$.

In [Bo1], we gave another, equivalent definition of conical limit point. Namely, we said that $x \in \Lambda$ is a conical limit point if and only if there is a wandering net, $(\gamma_n)_n$ in Γ, such that for all $y \in \Lambda \setminus \{x\}$, the ordered pairs $(\gamma_n x, \gamma_n y)$ lie in a compact subset of the space of distinct pairs of Λ (i.e. $\Lambda \times \Lambda$ minus the diagonal). We can assume that $(\gamma_n)_n$ is a collapsing net, and that $(\gamma_n x, \gamma_n y)$ converges on some pair (a, b) with $a \neq b$. Now, either $\gamma_n | \Lambda \setminus \{x\} \twoheadrightarrow b$ or $\gamma_n | \Lambda \setminus \{y\} \twoheadrightarrow a$. However, the latter cannot occur, since choosing any $z \in \Lambda \setminus \{x, y\}$, we would get $(\gamma_n x, \gamma_n z) \to (a, a)$. If we now fix any $c \in \Lambda \setminus \{a, b\}$, and let $x_n = \gamma_n^{-1} c$, we see that $x_n \to x$. We thus arrive at our original definition of a conical limit point. The converse statement is elementary.

The following is a standard result in the case of Kleinian groups. (A proof for 3-dimensionial Kleinian groups is given in [BeaM] or [Mas], and generalised to any dimension in [SuS].) I'm indebted to Pekka Tukia for suggesting a means of significantly simplifying my original argument.

Proposition 3.2 : *A conical limit point cannot be a parabolic fixed point.*

Proof : Suppose, to the contrary, that $p \in M$ is both. Thus, there is a parabolic, $\beta \in \Gamma$, with fixed point p. Moreover, there is a point, $q \in M$, and nets $(x_n)_n$ and $(\gamma_n)_n$ in $M \setminus \{x\}$ and Γ respectively, with $x_n \to p$ and with $\gamma_n(p, q, x_n) \to (a, b, c) \in \Theta^0(M)$. As discussed above, we have $\gamma_n | M \setminus \{p\} \twoheadrightarrow b$. We can suppose that $\gamma_n p \neq b$ for all n.

Now, fix for the moment some n, and consider the net $(\gamma_m^{-1} \gamma_n)_m$ (where m ranges over the same directed set). Now, $\gamma_m^{-1} | M \setminus \{b\} \twoheadrightarrow p$, and so $\gamma_m^{-1} \gamma_n | M \setminus \{\gamma_n^{-1} b\} \to p$. Since $\gamma_n^{-1} b \neq p$, we see, by Lemma 2.5, that $\gamma_m^{-1} \gamma_n$ is loxodromic for all sufficiently large m.

Now, let $\delta_n = \gamma_n \beta \gamma_n^{-1}$. We claim that the net $(\delta_n)_n$ is wandering. For suppose not. This means that there is some n such that the set of $m > n$ with $\delta_m = \delta_n$ is cofinal. ¿From the last paragraph, we can find some $m > n$ with $\delta_m = \delta_n$ and with $\gamma_m^{-1} \gamma_n$ loxodromic. Now, β commutes with $\gamma_m^{-1} \gamma_n$. But by Lemma 2.2, a parabolic cannot commute with a loxodromic. This contradiction shows that $(\delta_n)_n$ is wandering as claimed. We can thus assume that $(\delta_n)_n$ is a collapsing net.

Now, $\gamma_n p \to a$ and $\delta_n(\gamma_n p) = \gamma_n p \to a$. Also $\gamma_n q \to b$ and $\delta_n(\gamma_n q) = \gamma_n(\beta q) \to b$. We see that either $\delta_n | M \setminus \{a\} \twoheadrightarrow b$ or $\delta_n | M \setminus \{b\} \twoheadrightarrow a$. Either way, since $a \neq b$, we see, by Lemma 2.5, that δ_n is loxodromic for all sufficiently large n. But δ_n is a conjugate of β, and hence parabolic. We thus arrive at a contradiction. $\diamondsuit$

We have already observed that, if Γ is a uniform convergence group on a perfect compactum, then every point of M is a conical limit point. Thus, by Proposition 3.1, we see that every Γ-orbit is dense and so the action of Γ on M is minimal. Moreover by Proposition 3.2, there are no parabolics. In summary, we have shown:

Proposition 3.3 : *The action of a uniform convergence group is minimal and contains no parabolics.* $\diamondsuit$

This result is well known for hyperbolic groups acting on their boundary, and so Proposition 3.3 also follows from the result of [Bo6] that every uniform convergence group on a perfect compactum arises in this way. It would be nice to have purely dynamical

proofs of other results concerning hyperbolic groups, for example, the fact that each torsion subgroup is finite.

One can show [T3,Bo6] that if Γ is a convergence group acting on a perfect metrisable compactum, M, such that every point of M is a conical limit point, then Γ is a uniform convergence group. This gives another topological characterisation of hyperbolic groups. It also suggests a way one might attempt to describe relatively hyperbolic groups dynamically. As in the case of Kleinian groups, or groups acting on pinched Hadamard manifolds [BeaM,Bo1], one could say that Γ is (minimal) "geometrically finite" if every point of M is either a conical limit point or a bounded parabolic fixed point. (A parabolic fixed point, p, is "bounded" if the quotient of $M \setminus \{p\}$ by the stabiliser of p is compact.) (Here M takes the place of the limit set of a Kleinian group.) There may be other possibilities for defining geometrical finiteness, for example using the space of distinct triples. It's unclear whether they all give rise to same notion in this generality. However, in the case where M is assumed to be the boundary of a complete locally compact Gromov hyperbolic space, X, and where the action is assumed to be derived from a properly discontinuous isometric action on X, then it would seem that all sensible notions of geometrical finiteness are equivalent, and thus give rise to a well-defined notion of relatively hyperbolic group (as suggested in [Gr]). It is thus an interesting question as to whether every (dynamically defined) geometrically finite convergence group arises in this way.

4. Quasiconvex subgroups of uniform convergence groups.

In this section, we give a dynamical characterisation of quasiconvex subgroups of a uniform convergence group. Given that such groups are hyperbolic [Bo6], this will be seen to coincide with the usual geometrical notion, though we shall develop the ideas as far as possible without bringing such geometrical considerations into play. This dynamical formulation is relevant to the construction of canonical splittings of uniform convergence groups in [Bo3]. It will be an almost immediate consequence of the definition that a quasiconvex subgroup acts as a uniform convergence group on its limit set. We shall see that, in fact, the converse also holds, though the proof we give here is geometric, and requires the result of [Bo6]. It would be nice to have a dynamical argument. We also note, from this result, that every torsion subgroup is finite.

Let's suppose then that Γ acts as a uniform convergence group on a perfect compactum, M. We first note that a subgroup of Γ is elementary, in the sense defined in Section 2, if and only if it is finite or two-ended. It's natural to deem every finite and two-ended subgroup to be quasiconvex. This allows us to restrict attention to subgroups, G, which are non-elementary. Note that in this case, the limit set, ΛG, is perfect.

Suppose that $(F_\xi)_{\xi \in \Xi}$ is a collection of closed subsets of M indexed by some set Ξ. We say that $(F_\xi)_{\xi \in \Xi}$ is *discrete on distinct pairs* if whenever we have two disjoint closed subsets, K and L, of M, the set $\{\xi \in \Xi \mid F_\xi \cap K \neq \emptyset$ and $F_\xi \cap L \neq \emptyset\}$ is finite.

Suppose $G \leq \Gamma$ is non-elementary. Let $\Xi = \Gamma/G$ be the set of left cosets of G in Γ. Suppose $F \subseteq M$ is a closed G-invariant set. We can index the Γ-images of F by the set Ξ by setting $F_\xi = \gamma F$, where $\gamma \in \Gamma$ and $\xi = \gamma G$.

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Definition : We say that G is *quasiconvex* if there exists a nonempty closed G-invariant subset, $F \subseteq M$ such that the collection of Γ-images, $(F_\xi)_{\xi \in \Xi}$, is discrete on distinct pairs, where ξ ranges over the set, $\Xi = \Gamma/G$, of left cosets of G.

Note that, without loss of generality, we can take F to be the limit set, ΛG, of G.

One can give an equivalent, more intuitive definition of quasiconvexity as follows. Suppose $K \subseteq M$ is closed. Write $\Theta_M(K) = \{\{x, y, z\} \in \Theta(M) \mid x, y \in K\}$. Thus, $\Theta(K) \subseteq \Theta_M(K) \subseteq \Theta(M)$. Note that $\Theta_M(K)$ is a closed subset of $\Theta(M)$.

Suppose that $G \leq \Gamma$ is a non-elementary subgroup, and that $\Lambda \subseteq M$ is a non-empty closed G-invariant set. We see that G acts properly discontinuously on $\Theta_M(\Lambda)$. Let us suppose that $\Theta_M(\Lambda)/G$ is compact.

We first observe that Λ is perfect. To see this, we use the same argument that lead us to Proposition 3.1 to show that if $b \in \Lambda$, then there is some $a \in \Lambda$ and a net, $(\gamma_n)_n$, of elements of G with $\gamma_n|M \setminus \{a\} \twoheadrightarrow b$. (For this we use that fact that M is perfect.) Since G is non-elementary, we know that Λ contains at least 3 elements, and so we can choose some $c \in \Lambda \setminus \{a, b\}$. Now, $\gamma_n c \to b$, showing that b is not isolated in Λ.

Now, $\Theta(\Lambda)/G$ is a closed subset of $\Theta_M(\Lambda)/G$ and hence also compact. In other words, G acts as a uniform convergence group on Λ. Applying Proposition 3.3 in this case, we conclude that Λ is a minimal non-empty closed G-invariant set. In other words, Λ is precisely the limit set, ΛG, of G.

We claim that the above condition is equivalent to quasiconvexity as we have defined it.

Proposition 4.1 : *A non-elementary group, $G \leq \Gamma$ is quasiconvex if and only if there is a nonempty closed G-invariant set, $\Lambda \subseteq M$, such that $\Theta_M(\Lambda)/G$ is compact. In such a case, Λ is necessarily the limit set of G. Moreover, G acts a uniform convergence group on Λ.*

Proof : We have already observed that last two statements follow from the our second definition of quasiconvexity, so it remains to show these definitions are equivalent.

Suppose first, that we have a such a set Λ with $\Theta_M(\Lambda)/G$ compact. There is a compact set, $P \subseteq \Theta_M(\Lambda)$ with $\Theta_M(\Lambda) = \bigcup GP$.

Suppose that $K, L \subseteq M$ are disjoint closed subsets. We want to show that $\{\xi \in \Xi \mid \Lambda_\xi \cap K \neq \emptyset$ and $\Lambda_\xi \cap L \neq \emptyset\}$ is finite. Without loss of generality, we can suppose that $M \neq K \cup L$. We choose any $z \in M \setminus (K \cup L)$. Let $Q = \{\{x, y, z\} \in \Theta(M) \mid x \in K, y \in L\}$. Thus, Q is a compact subset of $\Theta(M)$.

Now, suppose that $\xi = \gamma G$ with $K \cap \gamma \Lambda \neq \emptyset$ and $L \cap \gamma \Lambda \neq \emptyset$. Choose any $x \in K \cap \gamma \Lambda$ and $y \in L \cap \gamma \Lambda$, and let $\theta = \{x, y, z\} \in Q$. Now, $\gamma^{-1}\theta \in \Theta_M(\Lambda)$ and so, without loss of generality, we can suppose that $\gamma^{-1}\theta \in P$. In other words, $Q \cap \gamma P \neq \emptyset$. Since Γ acts properly discontinuously on $\Theta(M)$, we see that there are only finitely many possibilities for γ and hence for ξ. This shows that G is quasiconvex by the original definition.

The converse is most conveniently expressed in terms of nets. Suppose $(\theta_n)_n$ is a net in $\Theta_M(\Lambda)$. Let $\theta_n = \{x_n, y_n, z_n\}$ with $x_n, y_n \in \Lambda$ and $z_n \in M$. Since $\Theta(M)/\Gamma$ is compact, we can find a subnet, $(\theta_i)_i$, and elements $\gamma_i \in \Gamma$ with $\gamma_i x_i \to x$, $\gamma_i y_i \to y$ and $\gamma_i z_i \to z$, where

$x, y, z \in M$ are distinct. Now, $\gamma_i x_i, \gamma_i y_i \in \gamma_i \Lambda$, and so, from the quasiconvexity hypothesis, there are only finitely many possibilities for the cosets $\gamma_i G$. Thus, after passing to a further subsequence, we can find some fixed $\gamma \in \Gamma$ with $\gamma^{-1} \gamma_i \in G$ for all i. But now, $\gamma^{-1} \gamma_i \theta_i$ converges in $\Theta_M(\Lambda)$. This shows that $\Theta_M(\Lambda)/G$ is compact as required. $\diamondsuit$

Note that if G is quasiconvex, then the setwise stabiliser of ΛG is also quasiconvex and contains G as a finite index subgroup. In fact this stabiliser is precisely the commensurator, $\mathrm{Comm}(G)$, of G in Γ, in other words the set of elements $\gamma \in \Gamma$ such that G and $\gamma G \gamma^{-1}$ are commensurable. In this case, $\mathrm{Comm}(G)$ is the unique maximal subgroup of Γ which contains G as a subgroup of finite index. We also note that quasiconvexity is a commensurability invariant.

We next show that the notion of quasiconvexity we have defined coincides with the standard geometrical one. For this we need to appeal to the fact that a uniform convergence group is hyperbolic [Bo6], and that the action is topologically conjugate to the action on the boundary.

Suppose that Γ is a hyperbolic group. Let (X, d) be a Cayley graph for Γ. We identify $\partial X = \partial \Gamma$. Let k be some constant greater than the hyperbolicity constant with respect to the "thin triangles" definition. Let V be the vertex set of X. Given $\theta \in \Theta(\partial \Gamma)$, let $V(\theta) \subseteq V$, be the set of vertices $a \in V$ such that there exist biinfinite geodesics, $\alpha_1, \alpha_2, \alpha_3$ connecting the three points of θ with $d(a, \alpha_i) \le k$ for each i. Thus, $V(\theta)$ is finite and non-empty. (We shall refer to an element of $V(\theta)$ as a *centre* for θ.) Given any subset, $F \subseteq \Theta(\partial \Gamma)$, write $V(F) = \bigcup_{\theta \in F} V(\theta)$. If F is compact, then again, $V(F)$ is finite. Note that $V(\Theta(\partial \Gamma)) = V$.

Suppose that $x, y \in \partial \Gamma$ are distinct. Let α be a biinfinite geodesic connecting x to y. It's easy to see that $V(\Theta_{\partial \Gamma}(\{x, y\}))$ lies in some uniform neighbourhood of α. Conversely, suppose $a \in \alpha$. Now, $a \in V(\theta)$ for some $\theta \in \Theta(\partial \Gamma)$. By a simple geometric argument, applied to the set $\theta \cup \{x, y\}$, we see that a lies a bounded distance from a point of $V(\{x, y, z\})$ for some $z \in \theta$. (This bound depends only on the hyperbolicity constant.) But, $\{x, y, z\} \in V(\Theta_{\partial \Gamma}(\{x, y\}))$. In other words we see that α lies inside a uniform neighbourhood of $V(\Theta_{\partial \Gamma}(\{x, y\}))$. In particular, this shows that $V(\Theta_{\partial \Gamma}(\{x, y\}))$ is quasiconvex in the geometric sense. Moreover the constant of quasiconvexity is a function only of the hyperbolicity constant.

More generally, suppose that $K \subseteq \partial \Gamma$ is closed. Now, $\Theta_{\partial \Gamma}(K) = \bigcup \{\Theta_{\partial \Gamma}(\{x, y\}) \mid x, y \in K, x \ne y\}$. Thus, $V(\Theta_{\partial \Gamma}(K))$ is a union of sets of the form $V(\Theta_{\partial \Gamma}(\{x, y\}))$, which we showed, in the last paragraph, to be uniformly quasiconvex. Now, x is an ideal point of $V(\Theta_{\partial \Gamma}(\{x, y\}))$. It follows that for any two sets in this collection there is a third which shares an ideal point with each. ¿From this, it's a simple geometric argument to see that their union is (geometrically) quasiconvex. We have shown:

Lemma 4.2 : *If $K \subseteq \partial \Gamma$ is closed, then $V(\Theta_{\partial \Gamma}(K))$ is (geometrically) quasiconvex.* $\diamondsuit$

A subgroup, G, of Γ is geometrically quasiconvex if the G-orbit of some (and hence every) point of V is quasiconvex, or equivalently if there is a G-invariant quasiconvex subset, $Q \subseteq V$, with Q/G finite.

We can now prove the equivalence of this with our dynamically defined notion.

Proposition 4.3 : *Suppose Γ is hyperbolic, and $G \leq \Gamma$. Then G is geometrically quasiconvex if and only if it is quasiconvex (by our earlier definition) with respect to the action of Γ on $\partial\Gamma$.*

Proof : First note that Γ contains no infinite torsion subgroup. Thus, every elementary subgroup of Γ is quasiconvex by either definition, so we can suppose that G in non-elementary.

Suppose, first, that $\Lambda \subseteq \partial\Gamma$ is a non-empty closed G-invariant subset with $\Theta_{\partial\Gamma}(\Lambda)/G$ compact. Thus, $V(\Theta_{\partial\Gamma}(\Lambda))/G$ is finite. By Lemma 4.2, $V(\Theta_{\partial\Gamma}(\Lambda))$ is quasiconvex. It follows that G is geometrically quasiconvex.

Conversely, suppose that G is geometrically quasiconvex. Let $Q \subseteq V$ be a G-invariant quasiconvex subset with Q/G finite. Note that the limit set, ΛG, is precisely the set of ideal points of Q. Choose any $a \in V$. Now it's easily seen that for any $r \geq 0$, the set of $\gamma \in \Gamma$ such that $d(\gamma^{-1}a, Q) = d(a, \gamma Q) \leq r$ lie in finitely many left cosets of G in Γ. ¿From this its easy to see that the collection of Γ-images of ΛG, indexed by the left cosets of G, is discrete on distinct pairs. Thus, G is quasiconvex by our original definition. $\diamondsuit$

We noted earlier that a quasiconvex subgroup acts as a uniform convergence group on its limit set. This suggest an alternative definition. One might define a subgroup, G, of Γ to be quasiconvex if it is elementary, or if there is a nonempty closed perfect G-invariant subset, $\Lambda \subseteq \partial\Gamma$, such that $\Theta(\Lambda)/G$ is compact. Again, Λ is necessarily the limit set of G, so this is the same as asserting that G acts as a uniform convergence group on its limit set. (We don't really need to assume that Λ is perfect in the definition, since any closed subset has a natural perfect closed subset, which is non-empty in the case where G is non-elementary.)

To show that this apparently weaker definition is equivalent to the standard one, suppose G is non-elementary, and that $\theta(\Lambda)/G$ is compact. Let $A = V(\Theta(\Lambda))$ and $B = V(\Theta_{\partial\Gamma}(\Lambda))$. We know that A/G is finite. If we can show that B lies inside a uniform neighbourhood of A, then it follows that B/G is finite, and so $\Theta_{\partial\Gamma}(\Lambda)/G$ is compact, as required.

Suppose, to the contrary, that there is a sequence $(b_i)_{i \in \mathbf{N}}$, of points of B with $d(b_i, A) \to \infty$. Let a_i be the nearest point of A to b_i. Since A/G is finite, we can suppose, after translating by elements of G, and passing to a subsequence, that $a_i = a$ is constant. Moreover, we can suppose that $(b_i)_i$ converges on some point $b \in \partial\Gamma$. Now, each b_i lies a bounded distance from a geodesic connecting a pair of points of Λ. A simple geometric argument shows that by choosing one element from each such pair, we can find a sequence of points of Λ tending to b. This shows that $b \in \Lambda$.

Now, since Λ is perfect, we can find a sequence, $(x_i)_i$, of points of $\Lambda \setminus \{b\}$ tending to b. Fix any point, $y \in \Lambda \setminus \{b\}$, and let c_i be a centre for the three points b, y, x_i. Thus $c_i \in A$, and $c_i \to b$. Moreover the points c_i all lie a bounded distance from a fixed geodesic (namely one connecting y to b). Since $b_i \to b$, a simple geometric argument shows that we can find i, j such that $d(b_i, a) > d(b_i, c_j)$, contradicting the fact that a is the nearest point of A to b_i. This gives the result.

I don't know of a purely dynamical proof of this in general. It's not hard to find such a proof in the case where Λ is connected, using the first definition we gave of quasiconvexity.

5. Compactifications.

In this section, we describe how uniform convergence actions naturally "compactify" properly discontinuous cocompact actions. The typical example is that of a hyperbolic group acting on its Cayley graph, X. We can compactify X as $X \cup \partial\Gamma$, and extend the action of Γ to a convergence action on this space. Moreover the induced action on $\partial\Gamma$ is a uniform convergence action.

Suppose that M is a perfect compactum admitting a uniform convergence action by some group Γ. We know from [Bo6] that Γ is hyperbolic, and that M is equivariantly homeomorphic to $\partial\Gamma$. (In [Bo6] we assume M to be metrisable. However, the arguments go through without this assumption, replacing sequences by nets. One can therefore conclude, in retrospect, that M must be metrisable.)

Lemma 5.1 : *If a group Γ acts as a uniform convergence group on perfect compacta, M and N, then there is a unique Γ-equivariant homeomorphism from M onto N.*

Proof : The existence of such a homeomorphism follows from [Bo6], as mentioned above. The uniqueness will be a corollary of Proposition 5.5, though one can give a direct argument as follows. Suppose that g and h are two such homeomorphisms, and that $x \in M$. Now x is a conical limit point, so it is the attracting point, in M, of a collapsing net, $(\gamma_n)_n$ in Γ. Thus, $(\gamma_n)_n = (g \circ \gamma_n \circ g^{-1})_n = (h \circ \gamma_n \circ h^{-1})_n$ is also a collapsing net for N with attracting point $g(x) = h(x)$. $\diamond$

This is the only point that we need to make any reference to the the result of [Bo6]. This can be avoided simply by taking the existence of such a homeomorphism as hypothesis where necessary.

We give a few definitions.

Definition : By a *compactified space*, $(X, \partial X)$, we mean a compactum, $X \cup \partial X$, with a partition into two disjoint subsets, X and ∂X, with X open and dense in $X \cup \partial X$ (so that ∂X is closed).

Note that if $(X, \partial X)$ is a compactified space, then X is locally compact hausdorff, and ∂X is a compactum.

Definition : A *morphism* $f : (Y, \partial Y) \longrightarrow (X, \partial X)$ between two compactified spaces consists of a continuous surjective map, $f : Y \cup \partial Y \longrightarrow X \cup \partial X$, such that $f(Y) = X$ and $f|\partial Y \longrightarrow \partial X$ is a homeomorphism.

Note that $f|Y : Y \longrightarrow X$ is a proper continuous surjection. Note also that the composition of morphisms is a morphism. This definition also gives a notion of isomorphism of compactified spaces, where the morphism is assumed to be invertible.

Suppose that $(X, \partial X)$ is a compactified space, and that the group Γ acts by homeomorphism on $X \cup \partial X$, respecting the partition into X and ∂X. In other words, Γ acts by isomorphism on the space $(X, \partial X)$.

Definition : We say that Γ acts *properly* on $(X, \partial X)$ if the action on $X \cup \partial X$ is a convergence action, the action on ∂X is a uniform convergence action, and the action on X is properly discontinuous and cocompact.

We see easily that ∂X is precisely the limit set of the action on $X \cup \partial X$. (In particular, the attracting and repelling point of any collapsing net lie in ∂X.)

Typical examples of such actions are that of a hyperbolic group, Γ, on $(X, \partial \Gamma)$, where X is the Cayley graph of the group; or that of a group on the space $(\Theta(M), \partial \Theta(M))$ in induced by a uniform convergence action on M.

We note:

Lemma 5.2 : *Suppose that Γ acts by isomorphism on compactified spaces $(X, \partial X)$ and $(Y, \partial Y)$, and that $f : (Y, \partial Y) \longrightarrow (X, \partial X)$ is a Γ-equivariant morphism. Then Γ acts properly on $(X, \partial X)$ if and only if it acts properly on $(Y, \partial Y)$.*

Proof : This result follows from Lemma 1.9 and Proposition 1.10, except for one point, namely that if Γ acts as a convergence group on $X \cup \partial X$ then it acts as a convergence group on $Y \cup \partial Y$. To see this, suppose that $(\gamma_n)_n$ is a wandering net in Γ. Passing to a subnet, we can suppose that $\gamma_n|(X \cup \partial X) \setminus \{a\} \twoheadrightarrow b$, where $a, b \in \partial X$. If $a', b' \in \partial Y$ are the preimages of $a, b \in \partial X$, the we see easily that $\gamma_n|(Y \cup \partial Y) \setminus \{a'\} \twoheadrightarrow b'$. $\diamond$

We want to consider the uniqueness of compactifications of properly discontinuous cocompact actions. A useful observation is the following:

Lemma 5.3 : *Suppose the group, Γ, acts properly on compactified spaces, $(X, \partial X)$ and $(X', \partial X')$. Suppose that $K \subseteq X$ and $K' \subseteq X'$ are compact subsets, and that $h : \partial X \longrightarrow \partial X'$ is a Γ-equivariant homeomorphism. Then, $\bigcup \Gamma(K \times K') \cup \mathrm{graph}(h)$ is a closed subset of $(X \cup \partial X) \times (X' \cup \partial X')$.*

Proof : Certainly, $\bigcup \Gamma(K \times K')$ and $\mathrm{graph}(h)$ are closed in $X \times X'$ and $\partial X \times \partial X'$ respectively. If the conclusion fails, we can find some point, (x, y), of $((X \cup \partial X) \times (X' \cup \partial X')) \setminus (X \times X)$ which lies in the closure of $\bigcup \Gamma(K \times K')$ and with $y \neq h(x)$. Without loss of generality, we can suppose that $x \in \partial X$ (otherwise replace h by h^{-1}). We can find nets, $(x_n)_n$, $(y_n)_n$ and $(\gamma_n)_n$ in K, K' and Γ respectively, such that $\gamma_n x_n \to x$ and $\gamma_n y_n \to y$. Passing to a subnet, we can suppose that $(\gamma_n)_n$ is a collapsing net for both $X \cup \partial X$ and $X' \cup \partial X'$, with attracting points $a \in \partial X$ and $b \in \partial X'$ respectively. Now, considering the action of Γ on $\partial X'$, we see that $h(a)$ is the attracting point of the collapsing net $(h \circ (\gamma_n|\partial X) \circ h^{-1})_n = (\gamma_n|\partial X')_n$. We see that $b = h(a)$. But now, returning to $X \cup \partial X$ and $X' \cup \partial X'$, we know that $\gamma_n|K$ and $\gamma_n|K'$ converge uniformly to a and $b = h(a)$ respectively. Thus, $\gamma_n x_n \to a$ and $\gamma_n y_n \to b$, so $x = a$ and $y = h(a)$. This gives us the contradiction that $y = h(x)$. $\diamond$

Corollary 5.4 : *Suppose the group Γ acts properly on the compactified spaces $(X, \partial X)$ and $(X', \partial X')$ (so that there is a Γ-equivariant homeomorphism from ∂X to $\partial X'$). Then,*

there is a compactified space, $(Y, \partial Y)$, admitting a proper Γ-action, and Γ-equivariant morphisms $f : (Y, \partial Y) \longrightarrow (X, \partial X)$ and $f' : (Y, \partial Y) \longrightarrow (X', \partial X')$.

Proof : Choose compact sets $K \subseteq X$ and $K' \subseteq X'$ such that $X = \bigcup \Gamma K$ and $X' = \bigcup \Gamma K'$. Let $h : \partial X \longrightarrow \partial X'$ be a Γ-equivariant homeomorphism. Let $Y = \bigcup \Gamma(K \times K')$ and let $\partial Y = \mathrm{graph}(h)$. By Lemma 5.3, we see that $Y \cup \partial Y$ is a closed subset of $(X \cup \partial X) \times (X' \cup \partial X')$. We see that $(Y, \partial Y)$ is a compactified space. Moreover, the coordinate projections, $f : Y \cup \partial Y \longrightarrow X \cup \partial X$ and $f' : Y \cup \partial Y \longrightarrow X' \cup \partial X'$ are morphisms. $\diamondsuit$

(In fact, it's not hard to see directly that the conclusion of Corollary 5.4 defines an equivalence relation on the proper actions of a fixed group. We have thus shown that, for any given group, there is just one equivalence class.)

Proposition 5.5 : Suppose that Γ acts properly on compactified spaces $(X, \partial X)$ and $(X', \partial X')$. Suppose that $f : X \cup \partial X \longrightarrow X' \cup \partial X'$ is a Γ-equivariant function with $f(X) \subseteq X'$ and $f|X$ continuous, and with $f|\partial X$ a homeomorphism onto $\partial X'$. Then, f is continuous.

Proof : Let $K \subseteq X$ be compact, with $X = \bigcup \Gamma K$. Let $Y = \bigcup \Gamma(K \times fK) \cup \mathrm{graph}(f|\partial X)$. Thus, $\mathrm{graph}(f) \subseteq Y$, and Y is a closed subset of $(X \cup \partial X) \times (X' \cup \partial X')$.

Suppose $x_n \to x \in \partial X$ and $f(x_n) \to y \in X' \cup \partial X'$. We see that $(x, y) \in Y \cap (\partial X \times (X' \cup \partial X')) = Y \cap (\partial X \times \partial X') = \mathrm{graph}(f|\partial X)$ and so $y = f(x)$. This shows that f is continuous on ∂X, and hence on $X \cup \partial X$. $\diamondsuit$

Thus, if $f|X$ is a homeomorphism of X onto X', we get an isomorphism of $(X, \partial X)$ to $(X', \partial X')$. In particular, this proves the uniqueness of compactifications of properly discontinuous cocompact actions.

As remarked earlier, it also gives another proof of the uniqueness of the topological conjugacy between two uniform convergence actions — consider the induced actions on the compactified spaces of triples.

We need to consider the question of existence of compactifications. To this end, we begin by observing that one can reconstruct compactified spaces as domains or ranges of morphisms. We first make a few general topological observations.

Suppose that M is a compactum. The topology on M is unique among comparable topologies in the sense that any strictly coarser topology will fail to be hausdorff and any strictly finer topology will fail to be compact. If N is another compactum, and $f : N \to M$ is a continuous surjective map, then the topology on M is determined as the quotient topology. In other words it is the coarsest topology such that f is continuous. Alternatively, it is the unique hausdorff topology such that f is continuous. Note that if $U \subseteq M$ then the subspace topology on U is the quotient of the subspace topology on $f^{-1}U$.

More generally, if X and Y are locally compact hausdorff spaces, and $f : Y \longrightarrow X$ is a continuous proper surjective map, then the topology on X is also determined as the quotient topology. To see this, note that f extends to a continuous surjective map from the one-point compactification of Y to the one-point compactification of X. The statement follows from the observations of the previous paragraph.

Suppose, now, we are given a compactified space, $(Y, \partial Y)$, and a continuous proper surjective map, $f : Y \longrightarrow X$, to a locally compact hausdorff space, X (so that the topology on X is the quotient topology from Y). We may extend f to a morphism as follows. We let $\partial X = \partial Y$ (as a set), and extend f to a map $f : Y \cup \partial Y \longrightarrow X \cup \partial X$ by setting $f | \partial Y$ to be the identity. We give $X \cup \partial X$ the quotient topology. Clearly $X \cup \partial X$ is compact, and since every point preimage in $Y \cup \partial Y$ is compact (given that our original map f was proper) we see that $X \cup \partial X$ is also hausdorff. We also see that $f | \partial X$ is a homeomorphism, and X is open in $X \cup \partial X$. Also the new (subspace) topology on X is the quotient topology from Y, and thus agrees with the original. Clearly, if Y is dense in $Y \cup \partial Y$, then X is dense in $X \cup \partial X$. We conclude that $X \cup \partial X$ is a compactified space, and $f : (Y, \partial Y) \longrightarrow (X, \partial X)$ is a morphism. Moreover, $(X, \partial X)$ is unique up to isomorphism.

Now, suppose we are given a compactified space, $(X, \partial X)$, and a continuous proper surjective map, $f : Y \longrightarrow X$. As before, we want to extend f to a morphism. Again, we set $\partial Y = \partial X$ (as a set) and extend f to $Y \cup \partial Y$ by taking $f | \partial Y$ to be the identity. We topologise $Y \cup \partial Y$ by taking as base all open subsets of Y together with all sets of the form $f^{-1}V$ as V varies over open subsets of $X \cup \partial X$. This collection is clearly closed under finite intersection, and is thus indeed a base for a topology. Moreover, f is continuous, and so $X \cup \partial X$ is hausdorff. Also $f | \partial Y$ is a homeomorphism, and the subspace topology on Y agrees with the original. We need to check that $Y \cup \partial Y$ is compact. To this end, suppose that $\mathcal{U}$ and $\mathcal{V}$ are collections of open subsets of Y and $X \cup \partial X$ respectively, such that $\mathcal{U} \cup \{f^{-1}V \mid V \in \mathcal{V}\}$ covers $Y \cup \partial Y$. Now, $\mathcal{V}$ covers ∂X, and so there is some finite subset $\mathcal{V}_0 \subseteq \mathcal{V}$ covers ∂X. Let $K \subseteq X \setminus \bigcup \mathcal{V}_0$. Thus, K is a compact subset of X. Since f is proper, $f^{-1}K$ is a compact subset of Y, and hence of $Y \cup \partial Y$. It is thus covered by a subset, $\mathcal{U}_0 \cup \{f^{-1}V \mid V \in \mathcal{V}_1\}$, where $\mathcal{U}_0 \subseteq \mathcal{U}$ and $\mathcal{V}_1 \subseteq \mathcal{V}$ are finite. It follows that $Y \cup \partial Y$ is covered by $\mathcal{U}_0 \cup \{f^{-1}V \mid V \in \mathcal{V}_0 \cup \mathcal{V}_1\}$. Thus $Y \cup \partial Y$ is compact. Note that if X is dense in $X \cup \partial X$, then Y is dense in $Y \cup \partial Y$. We have shown that $(Y, \partial Y)$ is a compactified space, and that $f : (Y, \partial Y) \longrightarrow (X, \partial X)$ is a morphism.

In fact, the construction of the last paragraph is natural up to isomorphism. Indeed, the topology on $Y \cup \partial Y$ is determined as the unique compact topology inducing the original topology on Y and such that f is continuous. To see this, suppose that $Y \cup \partial Y$ admits another topology with this property. It's clear that this topology must be finer that constructed above. However, if it were strictly finer, then it would fail to be compact.

Suppose now that Γ acts properly on the compactified space $(X, \partial X)$, and acts properly discontinuously and cocompactly on a locally compact hausdorff space, X'. We can compactify X' as a space $(X', \partial X')$, admitting a proper Γ-action, as follows.

Let $K \subseteq X$ and $K' \subseteq X'$ be compact sets such that $X = \bigcup \Gamma K$ and $X' = \bigcup \Gamma K'$. Let $Y = \bigcup \Gamma (K \times K') \subseteq X \times X'$, and let $f : Y \longrightarrow X$ and $f' : Y \longrightarrow X'$ be the natural projection maps. Now, Y is closed in $X \times X'$, and hence locally compact. Moreover, f and f' are proper and surjective. As described earlier, we can find a compactified space $(Y, \partial Y)$ and extend f to a morphism $f : (Y, \partial Y) \longrightarrow (X, \partial X)$. Since this construction is natural, we get a Γ-action on $(Y, \partial Y)$ such that the map f is Γ-equivariant. By Lemma 5.2, this action is proper. Similarly, we construct a compactified space $(X', \partial X')$ admitting a proper Γ-action, and extend f' to a Γ-equivariant morphism $f' : (Y, \partial Y) \longrightarrow (X', \partial X')$.

Now given any uniform convergence action of a group, Γ, on a perfect compactum,

M, we can take, as starting point for the above construction, the proper action of Γ on the compactified space $(\Theta(M), \partial\Theta(M))$. We conclude:

Proposition 5.6 : *Suppose we are given a uniform convergence action of a group Γ on a perfect compactum, M. Suppose Γ acts properly discontinuously cocompactly on a locally compact hausdorff space X. Then there is a natural compactification of X as a compactified space $(X, \partial X)$ admitting a proper action of the group Γ, extending the action of Γ on X. This compactification is unique up to isomorphism. Moreover, there is a unique Γ-equivariant homeomorphism of M onto ∂X.* $\diamondsuit$

Note that if M is disconnected, then Γ has more than one end. This follows, either from Proposition 5.6, (taking X to be the Cayley graph), or appealing to [Bo6] and the standard fact for hyperbolic groups. Thus, Stallings's theorem [St] tells us that Γ splits over a finite subgroup. Dunwoody's accessibility theorem [D1] then leads us naturally to considering uniform convergence actions on continua.

6. Configuration spaces in continua.

In this section, we give some general results relating to continua (connected compacta). Our main concern will be with connectedness properties of configuration spaces. For applications, this means spaces of triples, though for the most part, we have little reason to restrict to this case. ¿From our discussion we can deduce something about convergence groups acting on continua. We shall proceed here in a fairly general manner, given what seem to be some useful general observations along the way. (For some general discussion of the theory of continua, see, for example, [HY] and [Na].)

Suppose that M is any hausdorff topological space. Let $\Pi_n^0(M) \subseteq M^n$ be the open subset of distinct ordered n-tuples (i.e. n-tuples with no two entries equal). Let $\Pi_n(M)$ be the quotient space of $\Pi_n^0(M)$ under the action of the symmetric group on n letters which permutes the coordinates. We think of an element of $\Pi_n(M)$ as an unordered n-tuple, in other words a subset of M of cardinality n. We refer to the spaces $\Pi_n^0(M)$ and $\Pi_n(M)$ as "configuration spaces". Note that $\Theta^0(M) = \Pi_3^0(M)$ and $\Theta(M) = \Pi_3(M)$. Our main interest is in $\Theta(M)$, though since there is nothing very special about the number 3, we may as well proceed in greater generality.

Our first main result (Theorem 6.3) tells us that if M is connected, then so is $\Pi_n(M)$ for all n. Alhough it is of no direct relevance to the rest of the paper, it is also interesting to consider when $\Pi_n^0(M)$ is connected (in the case of metrisable continua). Finally, if M is a Peano continuum, we shall see that $\Pi_n(M)$ has only one end for $n \geq 2$.

All of this is clearly related to the manner in which M is separated by finite subsets. This seems to be an interesting question in itself. As a special case we have the "treelike" nature of the set of global cut points of a connected hausdorff space, as formulated in [W]. (See also [Bo2].) More generally, we have the following lemma, which seems to be quite useful, though I haven't found any mention of it in the literature.

First, recall that a *quasicomponent* of a topological space is an equivalence class under the equivalence relation defined by deeming to points to be equivalent if every clopen

(closed and open) subset containing one also contains the other. Quasicomponents are always closed (though not necessarily connected). Clearly, a space is connected if and only if it has precisely one quasicomponent. For further discussion, see [HY]. If M is a connected hausdorff space, $C \subseteq M$ is closed, and $x, y \in M \setminus C$, we say that C *separates x from y in M* if x and y lie in different quasicomponents of $M \setminus C$. In other words, we can write $M \setminus C$ as a disjoint union of two open sets $M \setminus C = O \sqcup U$ with $x \in O$ and $y \in U$.

It turns out that if $F \subseteq M$ is any finite subset, then the separation properties of (subsets of) F are determined by an embedding of F in a finite graph. To be more precise, we construct a finite graph, $G = G(M, F)$, with vertex set $V(G) = F$, by joining $x, y \in F$ by an edge if x and y lie in the same quasicomponent of $(M \setminus F) \cup \{x, y\}$.

Lemma 6.1 : *Suppose M is a connected hausdorff space, and $F \subseteq M$ is finite. Let $G = G(M, F)$ be the finite graph described above. If $C \subseteq F$ and $a, b \in F \setminus C$, then C separates a from b in M if and only if C separates a from b in G.*

Proof : Suppose a and b are connected by an edge in G. Then a and b lie in the same quasicomponent of $(M \setminus F) \cup \{a, b\}$, and hence in the same quasicomponent of $M \setminus C$. More generally, it follows that if a and b are connected by a path in $G \setminus C$, then they lie in the same quasicomponent of $M \setminus C$.

To prove the converse, suppose that we can write $F = A \sqcup B \sqcup C$, with $a \in A$ and $b \in B$, and such that no edge of G connects any point of A to any point of B.

Suppose $x \in A$ and $y \in B$. By the definition of a quasicomponent, we can write $(M \setminus F) \cup \{x, y\} = O \sqcup U$, with $x \in O$, $y \in U$ and O, U open. We write $O(x, y)$ for some such set O. Note that its closure, $\bar{O}(x, y)$, is contained in $(O(x, y) \cup F) \setminus \{y\}$.

Now, let $O(x) = \bigcap_{y \in B} O(x, y)$. Thus, $O(x)$ is open, $x \in O(x)$ and $B \cap O(x) = \emptyset$. Moreover, $\bar{O}(x) \subseteq \bigcap_{y \in B} \bar{O}(x, y) \subseteq (O(x) \cup F) \setminus B = O(x) \cup (F \setminus B)$.

Now let $O = \bigcup_{x \in A} O(x)$. Thus, O is open, $A \subseteq O$, and $B \cap O = \emptyset$. Moreover, $\bar{O} \subseteq O \cup (F \setminus B) = O \cup (A \cup C) = O \cup C$. Let $U = M \setminus (\bar{O} \cup C)$. Then, $M \setminus C = O \sqcup U$, with O, U open, $a \in O$ and $b \in U$. Thus, a and b lie in different quasicomponents of $M \setminus C$. $\diamondsuit$

Corollary 6.2 : $G(M, F)$ *is connected.*

Proof : Take $C = \emptyset$. $\diamondsuit$

Before applying this to configuration spaces, we show how it gives us the "treelike" nature of cut points in a connected hausdorff space, as alluded to earlier. This treelike structure is critical in obtaining splitting of one-ended hyperbolic group with non-locally-connected boundary, leading eventually to the conclusion that no such groups can exist.

Suppose that M is a connected hausdorff space. Given $x, y, z \in M$, we say that y lies *between* x and z if y separates x and z in M. This defines a ternary "betweenness" relation on M. It's known that this relation satisfies certain axioms introduced by Ward [W]. These axioms turn out to be equivalent to the following property. Suppose $F \subseteq M$ is any finite subset, then we can embed F in a finite tree, T, such that if $x, y, z \in F$, then y lies between x and z in M if and only if y lies between x and z in T. In other words, if

we restrict Lemma 6.1 to the case where card $C \leq 1$, we can suppose that our finite graph is a tree (except that F need not the the entire vertex set of the tree).

Deducing this fact from Lemma 6.1 is elementary graph theory. Suppose that G is any finite connected graph. A *block* in G is a maximal 2-vertex connected subgraph. (We consider a single edge to be 2-vertex connected.) Thus, two blocks intersect, if at all, in a common vertex. Let $T(G)$ be the bipartite graph whose vertex set is an abstract disjoint union of the vertex set of G and the set of blocks of G. An edge of $T(G)$ connects a vertex to a block if and only if the vertex lies in the block (in G). One verifies that $T(G)$ is a tree. Moreover, if x, y, z are vertices of G and hence also of $T(G)$, then y lies between x and z in $T(G)$ if and only of y separates x from z in G. Thus, starting with F as a finite subset of our space M, and setting $G = G(M, F)$, we see that the betweenness relations on F as a subset of M agree with those on F as a subset of $T(G)$. An alternative proof of the existence of such a tree is given in [Bo2]. The treelike structures arising from these axioms are analysed in that paper, and, from a somewhat different perspective, in [AdN].

We now return to the objective of studying configuration spaces. As before, M is a connected hausdorff space. The following observation will be useful.

Fix $n \geq 2$, and suppose that $C \subseteq M$ is a subset with $n - 1$ elements. The map $[x \mapsto C \cup \{x\}] : M \setminus C \longrightarrow \Pi_n(M)$ is continuous. In particular, we see that if a and b lie in the same quasicomponent of $M \setminus C$, then $C \cup \{a\}$ and $C \cup \{b\}$ lie in the same quasicomponent of $\Pi_n(M)$. We can now prove:

Theorem 6.3 :   *If M is a connected hausdorff space, and $n \geq 1$, then $\Pi_n(M)$ is connected.*

Proof : Suppose, first, that $F \subseteq M$ is a subset with $n + 1$ elements. Thus, each $x \in F$ gives us an element, $F \setminus \{x\}$, of $\Pi_n(M)$. Now, if $x, y \in F$ are connected by an edge of $G = G(M, F)$, it follows, from the definition of G and the observation immediately preceding the proof, that $F \setminus \{x\}$ and $F \setminus \{y\}$ lie in the same quasicomponent of $\Pi_n(M)$. Since G is connected (by Corollary 6.2), it follows that the elements, $F \setminus \{x\}$ lie in the same quasicomponent of $\Pi_n(M)$ for all $x \in F$.

Now, we can get from any set of n elements of M to any other by moving one element at a time. ¿From the previous paragraph, we see that each such move keeps us in the same quasicomponent of $\Pi_n(M)$. It follows that $\Pi_n(M)$ has only one quasicomponent, and is thus connected. $\diamondsuit$

Although we shall have no need of the result here, it is amusing to ask when the space of distinct ordered n-tuples is connected. If we restrict to the case where M is a metrisable continuum, we can give a complete answer to this question:

Proposition 6.4 : *Suppose M is metrisable continuum not homeomorphic to an interval or a circle, then $\Pi_n^0(M)$ is connected.*

Clearly, if M is a (non-degenerate) interval, then $\Pi_n(M)$ has precisely $n!$ components, whereas if M is a circle, it has $(n-1)!$ components. (In these cases, everything is locally connected, so components and quasicomponents agree.)

We shall only give a rough sketch of the argument here. To this end we shall need topological characterisations of circles and finite trees among metrisable continua. (By a "finite tree", we really mean the realisation of a finite simplicial tree). Recall that a *(global) cut point* in a continuum is a point whose complement is disconnected. The following result can be found in [Na]:

Lemma 6.5 : *A metrisable continuum is a finite tree if and only if it has finitely many non-cut points.* ◇

A characterisation of the circle is given in [HY]. Thus, M is homeomorphic to a circle of and only if the complement of any pair of distinct point of M is disconnected. We shall need a slight variation on this, as follows.

Given a continuum, M, define a 4-ary relation, δ, on M by saying that $\delta(x, y, z, w)$ holds if and only if the pair $\{x, z\}$ separates y from w. We say that M is *cyclically separated* if δ is a cyclic order. The following can be deduced from the result cited in the previous paragraph. We omit the proof.

Lemma 6.6 : *A cyclically separated metrisable continuum is homeomorphic to a circle.* ◇

Now, Lemma 6.1 effectively reduces Proposition 6.4 to a problem in graph theory. Suppose G is a finite connected graph, and I is a set with n elements. Consider the collection of injective maps into the vertex set, $V(G)$, of G. We say that two such maps, $f, g : I \longrightarrow V(G)$ are related by a *move* if there is some $i \in I$ such that $f(i)$ and $g(i)$ are adjacent, and $f|I \setminus \{i\} = g|I \setminus \{i\}$. Intuitively, we imagine placing counters labelled by the elements of I on distinct vertices of G. We can think of a move as sliding a counter labelled i form one vertex to an adjacent vacant vertex. Suppose $A \subseteq V(G)$ is a subset of n elements, and we have two functions f, g which are related by a finite sequence of moves, and with $f(I) = g(I) = A$. Then, $g \circ f^{-1}$ gives us a permutation of A. The set of permutations arising in this way defines a subgroup of the symmetric group on n letters which is well defined up to conjugacy, and independent of the choice of A. If this subgroup is the whole symmetric group, we say that G is *n-permutable*. One can ask which graphs have this property. The following result is no doubt far from optimal. We omit the proof.

Lemma 6.7 : *Given any $n \in \mathbf{N}$, there is some $k(n) \in \mathbf{N}$ such that if G is a finite connected graph with at least $k(n)$ non-cut vertices, then either G is a circle, or it is n-permutable.* ◇

It seems that $k(n) = n + 2$ will do the trick. I'm not sure about $k(n) = n + 1$. In any case, any number, $k(n)$ will serve for our purposes.

We are now ready to sketch the proof.

Proof of Proposition 6.4 : Suppose that M is a metrisable continuum, and that $\Pi_n^0(M)$ is not connected. It follows that if $F \subseteq M$ is any finite subset, then $G(M, F)$ is not n-permutable.

Now, it's easy to see that among finite trees, the only possibility for M is an interval.

We can thus suppose that M is not a finite tree. By Lemma 6.5, we can find a set $A \subseteq M$ of non-cut points, with $\text{card}(A) = k(n)$.

Now, suppose $B \subseteq M$ is any finite subset. The graph, $G(M, A \cup B)$ is not n-permutable. Also, G has at least $k(n)$ non-cut vertices, corresponding to the subset A. Thus, by Lemma 6.6, G is a circle. It follows that the 4-ary relation, δ restricted to $A \cup B$, and so in particular to B, is a cyclic order. Since B was arbitrary, it follows that δ is a cyclic order on M. (In fact, it suffices to verify this for all subsets B with $\text{card}(B) = 5$.) It follows that M is cyclically separated, and hence, by Lemma 6.5, homeomorphic to a circle. $\diamondsuit$

We now restrict attention to Peano continua (locally connected continua). We shall see that, in this case, $\Pi_n^0(M)$ has one end.

First, we recall a bit of general topology. A hausdorff space, X, is locally connected if every point has a base of connected neighbourhoods. ¿From this, it's easily seen that each connected component of each open subset is open. It follows that, in fact, every point of X has a base of open connected neighbourhoods. Moreover, every open subset of X is locally connected, and every component of X is also a quasicomponent.

Suppose that X is also locally compact. We say that X "has one end" if, for every compact set, $K \subseteq X$, there is another compact set, $L \subseteq X$, with $K \subseteq L$ and $X \setminus L$ connected. In fact, its enough to find a compact set L such that $X \setminus L$ lies inside a single component of $X \setminus K$. (Since this component is open, its complement will be a compact set containing K.)

If M is a Peano continuum, then $\Pi_n(M)$ is connected, locally connected and locally compact. (Connectedness follows from Theorem 6.3. To see local connectedness at a point $\{x_1, \ldots, x_n\} \in \Pi_n(M)$, choose arbitarily small, pairwise disjoint neighbourhoods, $U_i \ni x_i$. The product of these neighbourhoods in M^n projects to a connected neighbourhood of $\{x_1, \ldots, x_n\}$ in $\Pi_n(M)$.) In fact:

Proposition 6.8 : *If M is a Peano continuum, and $n \geq 2$, then $\Pi_n(M)$ has one end.*

We shall only give a proof of this in the case of real interest to us, namely when $n = 3$, in other words, for $\Theta(M) = \Pi_3(M)$. It will be seen that the general case follows by a similar but slightly more complicated argument. We shall thus content ourselves with:

Proposition 6.9 : *If M is a Peano continuum, then $\Theta(M)$ has one end.*

Proof : Suppose $K \subseteq \Theta(M)$ is compact. A simple compactness argument shows that we can find a finite open cover, $(U_j)_{j \in J}$, of M with the property that if $i, j \in J$, $x, y \in U_i$ and $z \in U_j$, then $\{x, y, z\} \notin K$. We can also find a finite refinement, $(O_i)_{i \in I}$, of this cover with the property that if $i, j, k \in I$ with $O_i \cap O_j \neq \emptyset$ and $O_j \cap O_k \neq \emptyset$, then there is some $l \in J$ such that $O_i \cup O_j \cup O_k \subseteq U_l$. Moreover, we can take all the O_i and U_j to be connected.

Given $i, j \in I$, let $\Theta_{ij} = \{\{x, y, z\} \in \Theta(M) \mid x, y \in O_i, z \in O_j\}$. Thus, $\Theta_{ii} = \Theta(O_i) \subseteq \Theta(M)$. Let $E = \bigcup_{i,j \in I} \Theta_{ij}$, and let $L = \Theta(M) \setminus E$. We see that L is compact, and $K \subseteq L$. We claim that E lies in a single component of $\Theta(M) \setminus K$.

Let N be the nerve of the cover $(O_i)_{i \in I}$. In other words, N is the graph with vertex set I, and with $i, j \in I$ connected by an edge if and only if $O_i \cap O_j \neq \emptyset$. We see that N is connected.

Now, for each $k \in J$, the space $\Theta(U_k) \subseteq \Theta(M)$ is connected (by Theorem 6.3), and so lies in some component of $\Theta(M) \setminus K$. Also, if $i, j \in I$ with $O_i \cap O_j \neq \emptyset$, then there is some $k \in J$ with $O_i \cup O_j \subseteq U_k$, so that $\Theta(O_i) \cup \Theta(O_j) \subseteq \Theta(U_k)$. Thus $\Theta(O_i)$ and $\Theta(O_j)$ lie in the same component of $\Theta(M) \setminus K$. Since N is connected, we see that all the sets $\Theta(O_i)$, and hence all $\Theta(U_k)$ lie in this same component. We call this component D. We want to show that $E \subseteq D$, in other words, $\Theta_{ij} \subseteq D$ for all $i, j \in I$.

Suppose then that $i, j \in I$. Let $d = d(i, j)$ be the combinatorial distance, in N, between i and j. We prove, by induction on d, that $\Theta_{ij} \subseteq D$.

Suppose first, that $d \leq 2$. Then there is some $l \in J$ with $O_i \cup O_j \subseteq U_l$. Thus, $\Theta_{ij} \subseteq \Theta(U_l) \subseteq D$, as claimed.

Now suppose that $d \geq 3$. There is some $k \in I$, adjacent to j, with $d(i, k) = d - 1$. Now $d(i, k) \geq 2$, and so $O_i \cap (O_j \cup O_k) = \emptyset$. Fix any distinct $x, y \in O_i$, and consider the map $[z \mapsto \{x, y, z\}] : O_j \cup O_k \longrightarrow \Theta(M)$. This is continuous, and its image misses K. Since $O_j \cap O_k \neq \emptyset$, the set $O_j \cup O_k$ is connected, and so $Q = \{\{x, y, z\} \in \Theta(M) \mid z \in O_j \cup O_k\}$ lies in some component of $\Theta(M) \setminus K$. Now, $Q \cap \Theta_{ij} \neq \emptyset$ and $Q \cap \Theta_{ik} \neq \emptyset$. By the induction hypothesis, we have $\Theta_{ik} \subseteq D$, so it follows that $\Theta_{ij} \subseteq D$.

It follows, by induction, that $\Theta_{ij} \subseteq D$ for all $i, j \in D$, and so $E \subseteq D$. This shows that $\Theta(M)$ has one end. $\diamond$

The results of this section can be used to give direct proofs of some facts concerning convergence actions on continua. For example it's easy to see that such a group must be finitely generated. Moreover, we have:

Proposition 6.10 : *Any group which acts as a uniform convergence group on a Peano continuum is one-ended.*

Of course, these results also follow from the fact that such groups are hyperbolic.

To give a direct argument, suppose that M is a continuum, and that Γ is acts as a uniform convergence group on M. Now, $\Theta(M)$ is locally compact hausdorff, and so we can find an open set $O \subseteq \Theta(M)$, whose closure, $\bar{O}$, is compact, and with $M = \bigcup \Gamma O$. Since the action of Γ on $\Theta(M)$ is properly discontinuous, the cover ΓO is locally finite. Let N be the graph with vertex set $V(N) = \Gamma$ and with two vertices, γ and γ' joined by and edge if $\gamma O \cap \gamma' O \neq \emptyset$. Thus N is a locally finite graph, on which Γ acts with finite quotient. Moreover, using Theorem 6.3, we see that N is connected. Thus, Γ is finitely generated.

Suppose now, that $F \subseteq \Gamma \equiv V(N)$ is a finite set of elements which separates the graph, N into two unbounded pieces. In other words, we can write $V(N) \setminus F = A_1 \sqcup A_2$ with A_1 and A_2 both infinite, and with no edge connecting any element of A_1 to any element of A_2. Let $U_i = \bigcup_{\gamma \in A_i} \gamma O$. Thus, U_i is open and not relatively compact in $\Theta(M)$. Also $U_1 \cap U_2 = \emptyset$ and $\Theta(M) \setminus (U_1 \cup U_2) \subseteq \bigcup_{\gamma \in F} \gamma \bar{O}$ is compact. It follows that $\Theta(M)$ has more that one end. If M is a Peano continuum, Proposition 6.9 tells us that this is impossible. We deduce that N and hence, by definition, Γ is one-ended. This proves Proposition 6.10.

This is only the start of the analysis of uniform convergence groups acting on Peano

continua. As mentioned earlier, one can derive a lot more information about the structure of M and Γ. For example, M has no global cut point, and there is a bound on the valencies of local cut points. As for Γ, one can derive most of the essential features of the JSJ decomposition, as introduced by Sela [Se], from this hypothesis alone. In particular, this describes all possible splittings of Γ over two-ended subgroups. Details of this procedure, and applications to hyperbolic groups are given in [Bo3].

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