Hyperbolic 3-manifolds and the geometry of the curve complex

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Abstract.

We give a brief survey of some recent work on 3-manifolds, notably towards proving Thurston's ending lamination conjecture. We describe some applications to the theory of surfaces and mapping class groups.

0. Introduction.

There has recently been a great deal of activity in 3-manifold theory, with announcements of proofs of three major conjectures. In this paper, we will focus on some of the work surrounding one of these, namely the ending lamination conjecture, a proof of which was announced by Minsky, Brock and Canary in 2002. This, and related work has unearthed an array of fascinating interconnections between the mapping class groups, Teichmüller theory and the geometry of 3-manifolds.

Much of this can be viewed in the context of geometric group theory. This subject has seen very rapid growth over the last twenty years or so, though of course, its antecedents can be traced back much earlier. Two major sources of inspiration have been 3-manifold theory and hyperbolic geometry. The work of Thurston in the late 1970s [Th1,Th2] brought these subjects much closer together, and the resulting activity was one of the factors in launching geometric group theory as a subject in its own right. The work of Gromov has been a major driving force in this. Particularly relevant here is his seminal paper on hyperbolic groups [Gr].

In this paper, we give a brief overview of some of this recent work. As an illustration, we shall offer an example of how hyperbolic 3-manifolds can be used to study an essentially combinatorial problem concerning the curve complex associated to a compact surface. This complex, introduced by Harvey around 1980, has many nice topological and geometric properties.

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1. Coarse geometry.

In this section, we briefly recall some of the fundamental notions of geometric group theory. The general idea is to understand the "large scale" geometry of a metric space. This is sometimes termed "coarse" geometry since the invariants will not in general respect small scale geometry or topology. A fairly general reference is [BriH]. (We remark that a related but somewhat different viewpoint on coarse geometry is bound up with the Novikov and Baum-Connes conjectures, see for example [Ro], though we shall not discuss these matters here.)

Let (X, d) be a metric space. A *(global) geodesic* in X is a path $\pi : I \longrightarrow X$ such that $d(\pi(t), \pi(u)) = |t - u|$ for all $t, u \in I$, where I is a real interval. Usually we will not worry about parametrisations and identify π with its image in X. We say X is a *geodesic* space if every pair of points are connected by a geodesic. Examples are complete riemannian manifolds, or graphs where each edge is deemed to have unit length. The following is a fundamental notion:

Definition : A function $f : (X, d) \longrightarrow (X', d')$ (not necessarily continuous) between geodesic spaces is a *quasi-isometry* if there are constants, $c_1 > 0$, $c_2, c_3, c_4, c_5 \geq 0$, such that for all $x, y \in X$,

$$c_1 d(x, y) - c_2 \leq d'(f(x), f(y)) \leq c_3 d(x, y) + c_4$$

and for all $y \in X'$, there exists $x \in X$, such that $d'(y, f(x)) \leq c_5$.

We say that X, X' are *quasi-isometric* and write $X \sim X'$, if there is some quasi-isometry between them. One verifies that this defines an equivalence on geodesic spaces.

If a group, Γ, acts properly discontinuously of a proper (i.e. complete and locally compact) geodesic space X, then Γ is finitely generated. A key observation is that if the same group also acts properly discontinuously cocompactly on another such X', then X and X' are (equivariantly) quasi-isometric.

If Γ is any finitely generated group, then any Cayley graph of Γ with respect to a finite generating set is an example of such a space, and is therefore well-defined up to quasi-isometry. As examples, we see that (the Cayley graph of) the group of integers $\mathbf{Z}$ is quasi-isometric to the real line; $\mathbf{Z} \oplus \mathbf{Z}$ to the euclidean plane; and any free group to a tree. The fundamental group, $\pi(\Sigma_g)$, of the closed orientable surface, Σ_g, of genus $g \geq 2$ is quasi-isometric to the hyperbolic plane. The last example follows from the fact that Σ_g admits a hyperbolic structure, and so $\pi(\Sigma_g)$ acts properly discontinuously cocompactly on its universal cover, the hyperbolic plane, $\mathbf{H}^2$.

The following notion was introduced by Gromov [Gr]:

Definition : A geodesic space, X, is *k-hyperbolic* if for any triangle consisting of three geodesics, $\sigma_1, \sigma_2, \sigma_3$, in X, cyclically connecting three points, then σ_3 lies in a k-neighbourhood of $\sigma_1 \cup \sigma_2$. We say that X is *(Gromov) hyperbolic* if it is k-hyperbolic for some $k \geq 0$.

Note, in particular, that any two geodesics with the same endpoints remain bounded distance apart.

Expositions of this notion of hyperbolicity can be found in [GhH], [CoDP], [Sho] and [Bo1].

It turns out that hyperbolicity is quasi-isometry invariant. It thus makes sense to talk about a "hyperbolic group". Note that $\mathbf{H}^2$ (and indeed, $\mathbf{H}^n$ for any dimension, n) is hyperbolic and so $\pi_1(\Sigma_g)$ is a hyperbolic group. Any tree is 0-hyperbolic and so any finitely generated free group is hyperbolic. However, the euclidean plane and hence $\mathbf{Z} \oplus \mathbf{Z}$ is not. Indeed one can show that no hyperbolic group can contain $\mathbf{Z} \oplus \mathbf{Z}$ as a subgroup.

We remark that there are related notions of CAT(0) and CAT(-1) spaces, where geodesic triangles are assumed to be at least as "thin", in the appropriate metric sense, as the corresponding "comparison triangles" in the euclidean and hyperbolic planes respectively. These are not, however, quasi-isometry invariant. CAT(-1) implies both CAT(0) and hyperbolic.

2. Mapping class groups.

Let Σ be a compact orientable surface of genus g with p boundary components, and let $\kappa(\Sigma) = 3g+p-4$. We shall assume that $\kappa(\Sigma) > 0$. In other words, we are ruling out a small number of "exceptional" surfaces that can be independently understood. The *mapping class group*, $\mathrm{Map} = \mathrm{Map}(\Sigma)$, is the group of orientation preserving self-homeomorphisms of Σ defined up to homotopy. This group is finitely generated, but not hyperbolic: it has lots of $\mathbf{Z} \oplus \mathbf{Z}$ subgroups generated by pairs of disjoint Dehn twists (i.e. a pair of non-trivial mapping classes supported on disjoint annuli). The large scale geometry of (any Cayley graph of) Map has been studied by a number of authors, see for example [Ham].

In [Harv], Harvey associated a simplicial complex, $\mathcal{C} = \mathcal{C}(\Sigma)$ to Σ. Its vertex set, $V(\mathcal{C})$, is the set of homotopy classes of simple closed curves in $\mathcal{C}$ that cannot be homotoped to a point or to a boundary component of Σ. A subset, $A \subseteq V(\mathcal{C})$ is deemed to be a simplex if its elements can be realised disjointly in Σ. This complex is connected and has dimension $\kappa(\Sigma)$. We see that Map acts simplicially on $\mathcal{C}(\Sigma)$, pulling back curves under the homeomorphism, and that the quotient space is compact. The space $\mathcal{C}(\Sigma)$ is commonly referred to as the *curve complex* (or *Harvey complex*). We shall refer to its 1-skeleton, $\mathcal{G}(\Sigma)$, as the *curve graph*.

The curve complex has nice topological and combinatorial properties that can be used to study $\mathrm{Map}(\Sigma)$. For example, in [Hare], Harer investigates the cohomology of Map and in [Iv], Ivanov studies its automorphisms.

The Teichmüller space, $\mathcal{T} = \mathcal{T}(\Sigma)$, of Σ is the space of marked hyperbolic structures on the interior, $\mathrm{int}(\Sigma)$, of Σ. More precisely, an element of $\mathcal{T}$ consists of a complete finite-area hyperbolic surface, S, which is "marked" by a homotopy class of homeomorphisms, $\mathrm{int}(\Sigma) \longrightarrow S$. We see that Map acts on $\mathcal{T}$ by changing the marking. The quotient, $\mathcal{T}/\mathrm{Map}$

is the "moduli space" of unmarked hyperbolic structures. By uniformisation, studying hyperbolic structures is equivalent to studying conformal structures, that is, (punctured) Riemann surfaces.

The Teichmüller space has a very rich structure (see [ImT]). For example it is a complex manifold, and carries two, rather different, natural metrics, namely the Teichmüller metric and the Weil-Petersson metric. It is worth noting however, that:

Proposition 2.1 : *If d is any complete Gromov hyperbolic* Map*-invariant metric, then the action of* Map *on $\mathcal{T}$ must be parabolic (i.e. it fixes a unique point in the ideal boundary).*

This follows from an argument that is most easily expressible in terms of "convergence groups", as introduced by Gehring and Martin. In the above situation, Map would act as a convergence group on the ideal boundary. We have observed that any pair of disjoint Dehn twists generate a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of Map, which must be parabolic (see for example [Tu]). It follows that any Dehn twist fixes a unique ideal point, and since the curve graph is connected, these fixed points are all equal. The result now follows from the fact that Map can be generated by a set of Dehn twists. (Indeed any convergence group action of Map must fix a unique point.)

This effectively says that $\mathcal{T}$ admits no interesting invariant complete Gromov hyperbolic metric.

Topologically, $\mathcal{T}$, is an open $(6g - 6 + 2p)$-dimensional ball that can be naturally compactified to a closed ball by adjoining the space, $\partial\mathcal{T}$, of "projective laminations". This is the "Thurston compactification" [Bon].

Given $\alpha \in V(\mathcal{C})$ and $\epsilon > 0$, we write $\mathcal{T}_\epsilon(\alpha) \subseteq \mathcal{T}$ for the set of surfaces in which α can be realised as a curve of length less than ϵ. If $\epsilon = \epsilon(\Sigma) > 0$ is sufficiently small, then $A \subseteq V(\mathcal{C})$ is a simplex if and only if $\bigcap_{\alpha \in A} \mathcal{T}_\epsilon(\alpha) \neq \emptyset$. In other words, we can think of a $\mathcal{C}$ as a nerve to the family $(\mathcal{T}_\epsilon(\alpha))_{\alpha \in V(\mathcal{C})}$. Up to quasi-isometry, we can equivalently think of $\mathcal{C}$ as arising by "shrinking down" each $\mathcal{T}_\epsilon(\alpha)$ to a set of bounded diameter (starting, for example, with the Teichmüller metric on $\mathcal{T}$). We refer to $\bigcup_{\alpha \in V(\mathcal{C})} \mathcal{T}_\epsilon(\alpha)$ as the *thin part* of $\mathcal{T}$, and to its complement as the *thick part*. It is well known, following work of Mumford, that thick$(\mathcal{T})$/Map is compact (see for example [Ab]). Moreover, thick$(\mathcal{T})$ is connected, and we see that Map is equivariantly quasi-isometric to any invariant geodesic metric on thick$(\mathcal{T})$. In this way, we can also view $\mathcal{C}$ up to quasi-isometry as arising by shrinking down each of a family of subgroups of Map, namely the stablisers of simple closed curves.

In view of the fact that neither $\mathcal{T}$ nor thick$(\mathcal{T}) \sim$ Map admits a (sensible) proper invariant hyperbolic metric, the following result is striking:

Theorem 2.2 : [MasM1] *The curve complex, $\mathcal{C}$, is Gromov hyperbolic.*

Note that it is enough here to consider the curve graph, $\mathcal{G}(\Sigma)$, since its inclusion into $\mathcal{C}$ is a quasi-isometry.

A somewhat shorter proof can be found in [Bow3], which shows, in fact that the hyperbolicity constant is $O(\log \kappa(\Sigma))$.

A major complication in applying the usual machinery of hyperbolic groups to the curve graph arises from the fact that $\mathcal{G}$ is far from being locally finite. One way of dealing with this is suggested by Bestvina and Fujiwara [BeF], where they show that the action of Map on $\mathcal{G}$ is what they call "weakly properly discontinuous". As a result, they deduce:

Theorem 2.3 : [BeF] *The second bounded cohomology of* Map *is infinitely generated.*

Indeed, they deduce that the same holds for "most" subgroups of Map.

Here is another result concerning the action of Map on $\mathcal{G}$.

Theorem 2.4 : [Bow6]

(1) The action of Map *on* $\mathcal{G}$ *is acylindrical.*

(2) There is some $N = N(\Sigma) \in \mathbf{N}$ *such that for all* $g \in \mathrm{Map}$, $N||g|| \in \mathbf{N}$.

"Acylindricity" says essentially that there is a bound on the number of elements that can displace a long geodesics a short distance. (To be precise, for all $r \geq 0$, there exist $R, K \geq 0$ such that if $x, y \in V(\mathcal{G})$ with $d(x,y) \geq R$, then $|\{g \in \mathrm{Map} \mid d(x,gx) \leq r,\ d(y,gy) \leq r\}| \leq K$.) It is a natural property of an action on a hyperbolic space. In particular, it implies weak proper discontinuity in the sense of [BeF]. The *stable length*, $||g||$, of $g \in \mathrm{Map}$ is defined as $\lim_{n\to\infty} \frac{1}{n} d(x, g^n x)$ for any $x \in \mathcal{G}$. We are thus claiming that this is uniformly rational. The analogues of (1) and (2) above are known for hyperbolic groups. The proof of Theorem 4.4 will use hyperbolic 3-manifolds, and we say more about it in Section 4.

We conclude this section with some remarks about the Teichmüller and Weil-Petersson metrics.

The Teichmüller metric, d_T, is a complete geodesic Finsler metric. As we have observed, it cannot be hyperbolic, nor is it CAT(0) [Mas]. However, Teichmüller geodesics have a nice geometric description. For simplicity consider the case where Σ closed. A geodesic path $\pi : I \longrightarrow \mathcal{T}$ gives rise to a particular kind of singular riemannian metric, namely a "singular sol" geometry on $\Sigma \times I$, which we denote by P_π. If $\pi(I) \subseteq \mathrm{thick}(\mathcal{T})$, then the universal cover $\tilde{P}_\pi$ is Gromov hyperbolic. More generally, if $\sigma : I \longrightarrow \mathrm{thick}(\mathcal{T})$ is any path, we can construct a space $P_\sigma \cong \Sigma \times I$, essentially by assembling the hyperbolic surfaces $\sigma(t)$ for $t \in I$. Provided this is done in a reasonably sensible manner, the universal cover, $\tilde{P}_\pi$, is well defined up to $\pi_1(\Sigma)$-equivariant quasi-isometry. It follows from independent work in [Mo] and [Bow2] that:

Theorem 2.5 : *A path* $\sigma : I \longrightarrow \mathrm{thick}(\mathcal{T})$ *remains a bounded distance from a Teichmüller geodesic if and only if* $\tilde{P}_\sigma$ *is Gromov hyperbolic.*

(Of course one needs to interpret this in term of the uniformity of the various constants involved.)

The Weil-Petersson metric is rather different. It is a negatively curved riemannian Kahler metric. It is not complete, but nevertheless geodesic and globally CAT(0), see [W1,W2]. It is shown in [Bro] that $(\mathcal{T}, d_W)$ is quasi-isometric to the "pants complex",

$\mathcal{P} = \mathcal{P}(\Sigma)$ of Σ. This is a 2-dimensional cell complex related to the curve complex. Like the curve complex, up to quasi-isometry it can be thought of as obtained by shrinking down some (but this time not all) of the thin part of Teichmüller space, or as shrinking down certain subgroups of Map. In this way, its coarse geometry can be viewed as intermediate between those of Map and $\mathcal{C}$. It turns out that $\mathcal{P}$ is not hyperbolic except when Σ is a five-holed sphere or two-holed torus [BroF], and so the same follows for $(\mathcal{T}, d_W)$. See also [Ar] for a discussion of the exceptional cases. Some connections between the Weil-Petersson metric and hyperbolic 3-manifolds are discussed in [Bro].

In summary, we have seen four very natural quasi-isometry classes of metrics on which Map acts, namely Map $\sim$ thick$(\mathcal{T})$, $\mathcal{C}(\Sigma)$, $(\mathcal{T}, d_T)$ and $(\mathcal{T}, d_W) \sim \mathcal{P}(\Sigma)$. Each has some nice property not shared by any of the others, and understanding their interconnections is an intriguing problem.

3. 3-manifolds.

Two aspects of 3-dimensional space provide us with powerful tools in this dimension. The first arises from the fact that hyperbolic 3-space, $\mathbf{H}^3$ is naturally compactified to a ball by adjoining the Riemann sphere, $\mathbf{C} \cup \{\infty\}$, so that hyperbolic isometries correspond to conformal automorphisms. This gives rise to a rich analytic theory. The second stems from the topological theory of 3-manifolds developed over the last century. Such connections began to be exploited in the 1960s and 1970s, see for example [Mar], and the subject saw a revolution in the late 1970s arising out of the work of Thurston [Th1,Th2]. He proposed a number of conjectures. Among the most significant are:

(1) Geometrisation.

This says that any compact 3-manifold can be canonically cut into pieces each admitting a geometric structure — the main issues arising out of spherical and hyperbolic geometry. The topological decomposition alluded to had already been described in earlier work of Kneser and Milnor, and Waldhausen, Johanson, Jaco and Shalen. It should be noted that this work has served as a major source of inspiration in geometric group theory. We note, in particular, the splitting theory developed by Stallings, Dunwoody, Rips and many others as well as the more recent JSJ decomposition of Sela [Se] which is central to his work on the Tarski problem, and in which the mapping class groups of surfaces feature prominently.

Thurston proved many special cases of the geometrisation conjecture [O1,K]. Recently Perelman announced a proof in general [P1,P2]. This, of course, implies the famous Poincaré conjecture.

(2) Tameness.

This can be conveniently phrased as follows. If M is a complete hyperbolic 3-manifold with $\pi_1(M)$ finitely generated, then M is tame (or topologically finite), i.e. homeomorphic to the interior of a compact manifold. In fact, in this form, the conjecture is due to Marden [Mar]. Thurston gave a geometric reinterpretation which was later shown to be equivalent by Canary [Can]. Significant advance was made by Bonahon [Bon], and the general case was recently announced independently by Agol [Ag] and Calegari and Gabai [CalG].

(3) The ending lamination conjecture.

Suppose that M is a tame hyperbolic 3-manifold. The ending lamination conjecture (ELC) asserts that M is determined up to isometry by its topology together with a finite set of "end invariants". Work towards this conjecture has formed a major project of Minsky, along with coworkers, notably Masur. A general proof has now been announced in joint work with Brock and Canary [Mi4,BroCM]. See [Mi3] for a general survey.

For simplicity of exposition, consider the case where M has no cusps. Each end of M is of one of two types. It may be "geometrically finite", in which case it opens out exponentially fast, and can be naturally compactified by adjoining a Riemann surface (arising out of the identification of the boundary of $\mathbf{H}^3$ with $\mathbf{C} \cup \{\infty\}$). In the other "simply degenerate" case, the geometry is quite different. For example in the "bounded geometry" situation (see Section 4) the end is quasi-isometric to a ray $[0, \infty)$. The end invariant of a geometrically finite end is a point of $\mathcal{T}$, namely the compactifying Riemann surface. That of a simply degenerate end is a lamination, which (modulo forgetting about transverse measures) might be thought of as a point in $\partial \mathcal{T}$.

Suppose M_1 and M_2 are tame hyperbolic 3-manifolds, with the same topology and the same end invariants. Let $\Gamma = \pi_1(M_1) = \pi_1(M_2)$. We get actions of Γ on the universal covers $\tilde{M}_1$ and $\tilde{M}_2$, which are each isometric to $\mathbf{H}^3$. To prove the ELC, it turns out to be sufficient to find an equivariant quasi-isometry between their covers. This follows from the deformation theory of Kleinian groups developed by Ahlfors, Bers, Marden, Maskit and Sullivan, see for example, [K]. The geometrically finite case is already encompassed by this earlier work.

Since this all boils down to understanding the geometry of a (simply degenerate) end which we know to be homeomorphic to a surface times $[0, \infty)$, we can see most of the essential ideas just by considering surface groups.

4. Surface groups.

For simplicity, we consider only the closed surface case. Let $\Sigma = \Sigma_g$ be the closed orientable surface of genus $g \geq 2$, and let $\Gamma = \pi_1(\Sigma)$. Suppose that Γ acts properly discontinuously on $\mathbf{H}^3$ preserving orientation and without parabolics. Thus, $M = \mathbf{H}^3/\Gamma$ is a 3-manifold without cusps. In this case, tameness follows from [Bon], and so M is homeomorphic to $\Sigma \times \mathbf{R}$. Simply hyperbolic geometry tells us that any curve $\alpha \in V(\mathcal{G})$ can be uniquely realised as a closed geodesic $\bar{\alpha}$ in M. (Here we mean in the usual riemannian sense — it is only locally geodesic in the metric space sense defined earlier.)

We begin by recalling some of the standard Thurston machinery (see [CanEG]). By a *pleated surface* we shall mean a map $\phi : (\Sigma, \rho) \longrightarrow M$ which is homotopic to the inclusion of Σ in $M \cong \Sigma \times \mathbf{R}$, and which is 1-lipschitz with respect to some hyperbolic metric, ρ, on Σ. (Normally, pleated surfaces are assumed to be folded in a particular way, but all we require here is the lipschitz property. Indeed it would be enough for them to be uniformly lipschitz.) The hyperbolic structure, ρ, is viewed as part of the data of the pleated surface. In general, pleated surfaces are not embedded.

We say that ϕ *realises* a curve $\alpha \in V(\mathcal{G})$ if $\phi|\hat{\alpha}$ is a locally isometric map to $\bar{\alpha}$, where $\hat{\alpha}$ is the unique closed geodesic in (Σ, ρ) in the class of α. A relatively simple construction of [Th1] or of [Bon] shows:

Lemma 4.1 : *Any $\alpha \in V(\mathcal{G})$ can be realised by a pleated surface. Indeed, if $\alpha, \beta \in V(\mathcal{G})$ are adjacent then they can be realised by the same pleated surface.*

We see a connection with the curve graph emerging, since if $\gamma_0, \ldots, \gamma_n$ is any path in $\mathcal{G}$, we get a sequence of interlocking pleated surfaces, $\phi_i : (\Sigma, \rho_i) \longrightarrow M$, for $i = 1, \ldots, n$, where ϕ_i realises both γ_i and γ_{i-1}.

Now any sequence of curves $(\gamma_i)_{i=0}^n$ in $V(\mathcal{G})$ contains a subsequence converging on a lamination λ. This means that they can be realised in Σ so that they converge in the Hausdorff sense. Generically, a lamination is locally homeomorphic to a cantor set times an interval, though in general a transversal may also contain (or indeed consist entirely of) isolated points. A lamination thus consists of a set of 1-dimensional leaves foliating a closed subset of Σ. (If we were to fix a hyperbolic structure on Σ, we could realise this so that all leaves are riemannian geodesics.)

Suppose the end $e \equiv \Sigma \times [0, \infty)$ of M is simply degenerate. By [Bon], we get a sequence, $(\gamma_i)_{i=1}^\infty$ in $V(\mathcal{G})$ so that the realisations, $\bar{\gamma}_i$, go out the end e. Moreover, γ_i converges on a well defined lamination — the *ending lamination* of e (at least modulo removing isolated leaves from the limit).

We can also think of this in terms of Teichmüller space. We get sequence of pleated surfaces, $\phi_i : (\Sigma, \rho_i) \longrightarrow M$ realising γ_i. The images $\phi_i(\Sigma)$ also go out e. In the Thurston compactification, $\mathcal{T} \cup \partial\mathcal{T}$, of Teichmüller space, (Σ, ρ_i) converges on λ (at least after we have identified all projective laminations with support λ.)

In fact one can interpolate so that the γ_i are the vertices of an infinite ray in $\mathcal{G}(\Sigma)$, and this way get a sequence of interlocking pleated surfaces. (Indeed it follows from work of Minsky that one can take this ray to be geodesic in $\mathcal{G}$.)

The general strategy for proving the ELC is to construct a "model" metric on $\Sigma \times [0, \infty)$, depending only on the ending lamination λ, and then show that the universal covers of e and of the model space are Γ-equivariantly quasi-isometric.

A special case of the ELC is that of bounded geometry, i.e. where e has positive injectivity radius. It then follows that the images of all pleated surfaces in e have bounded diameter. This case is treated in [Mi1,Mi2], and one can take the model space to be the singular sol manifold P_π, where π is a geodesic ray in $\mathcal{T}$ tending to λ. In fact, by interpolating between the pleated surfaces in M, we get a path $\sigma : I \longrightarrow \mathrm{thick}(\mathcal{T})$ such that $\tilde{P}_\sigma$ is equivariantly quasi-isometric to the universal cover, $\tilde{e}$. One can deduce that P_σ is Gromov hyperbolic, and using Theorem 2.5, one sees that σ remains close to π, from which one deduces, in turn, that $\tilde{P}_\sigma$ is equivariantly quasi-isometric to $\tilde{P}_\pi$. In other words, one recovers the following result of Minsky:

Theorem 4.2 : *If the end e has bounded geometry, then $\tilde{e}$ is equivariantly quasi-isometric to the singular sol model space, $\tilde{P}_{\pi}$.*

We deduce the ELC in the bounded geometry case.

Unfortunately, Theorem 4.2 will certainly fail when we move away from bounded geometry (though a possible variant of this construction is proposed in [Re]).

In the general (indeed generic) case, e will contain arbitrarily short closed geodesics, which are necessarily simple [O2], and hence have the form $\bar{\gamma}$ where $\gamma \in V(\mathcal{G})$. Any path of pleated surfaces going out the end will inevitably have to pass through the corresponding thin parts, $\mathcal{T}_{\epsilon}(\gamma)$, of Teichmüller space. The picture can get very complicated, but the curve graph, $\mathcal{G}(\Sigma)$, offers a means of coming to terms with the situation. This was one of the motivations behind the study of [MasM1,MasM2]. The idea in [Mi4] is to construct a model space out of combinatorial data of the curve graph. The details are quite involved, but a key idea is that of a "tight" geodesic. (To interpret the following discussion correctly one should substitute "multicurve" for "curve", allowing a curve to have more than one component. However, we can safely ignore this somewhat tedious complication here.)

Let $(\gamma_i)_{i=0}^{n}$ be a geodesic in $\mathcal{G}$. We say that $(\gamma_i)_i$ is *tight* at γ_i if each curve that crosses γ_i also crosses either γ_{i-1} or γ_{i+1}. We say $(\gamma_i)_i$ is *tight* if is tight at γ_i for all $i = 1, \ldots, n-1$. Note that γ_i must be disjoint from the connected set $\gamma_{i-1} \cup \gamma_{i+1} \subseteq \Sigma$. In general, there may be infinitely many ways of choosing γ_i. Tightness obliges us to take one of the curves bounding the subsurface of Σ filled by $\gamma_{i-1} \cup \gamma_{i+1}$.

Let $T(\alpha, \beta)$ be the set of all tight geodesics from α to β in $\mathcal{G}$.

Theorem 4.3 : [MasM2]
(1) $T(\alpha, \beta)$ is nonempty.
(2) $T(\alpha, \beta)$ is finite.

(It is part (1) which seems to require us to reinterpret tightness in terms of multicurves.)

Given $r \in \mathbf{N}$, let $S_r(\alpha, \beta) = \{\gamma \in \bigcup T(\alpha, \beta) \mid d(\alpha, \gamma) = r\}$. In other words it is a "slice" through the union of all tight geodesics a given distance from one endpoint. We can refine Theorem 4.3(2) as:

Theorem 4.4 : [Bow5] *There is some $K = K(\text{genus}(\Sigma)) \in \mathbf{N}$ such that given any $\alpha, \beta \in V(\mathcal{G})$ and $r \in \mathbf{N}$, $|S_r(\alpha, \beta)| \leq K$.*

Note that the hyperbolicity of $\mathcal{G}$ tells us immediately that slices have bounded diameter. Theorem 4.4 states that they have bounded cardinality. In fact, there are refinements of this result that allow us to vary α and β, each within a set of bounded diameter, while retaining a cardinality bound on slices that remain far enough away from the endpoints.

One consequence of Theorem 4.4 (and its refinements) is that, for certain purposes, it effectively reduces us to considering locally finite graphs. In this way, a diagonal sequence argument, together with an argument of Delzant [D] in the context of hyperbolic groups, gives us:

Proposition 4.5 : *If $g \in$ Map and $||g|| > 0$, then there is a bi-infinite geodesic, $\pi \subseteq \mathcal{G}$, such that $g^N \pi = \pi$, where $N = N(\Sigma)$ depends only on the topological type of Σ.*

Thus, g^N translates π some distance $p \in \mathbf{N}$, and so $N||g|| = ||g^N|| = p \in \mathbf{N}$, proving Theorem 2.4(2). We remark that $||g|| > 0$ if and only if g is a pseudoanosov mapping class in the Nielsen-Thurston classification.

One can similarly use Theorem 4.4 to prove Theorem 2.4(1).

The proof of Theorem 4.4 uses the following relatively classical fact about hyperbolic 3-manifolds:

Lemma 4.6 : *Given any $\alpha, \beta \in V(\mathcal{G})$, we can find a complete hyperbolic 3-manifold, $M \cong \Sigma \times \mathbf{R}$, in which $\bar{\alpha}$ and $\bar{\beta}$ both have uniformly bounded length (indeed can be chosen arbitrarily short).*

Here we see the necessity of passing to 3 dimensions — there is no hope of achieving such a result for hyperbolic surfaces.

We need, in addition, the following:

Theorem 4.7 : *If $\alpha = \gamma_0, \ldots, \gamma_n = \beta$ is a tight geodesic with the lengths of $\bar{\alpha}$ and $\bar{\beta}$ uniformly bounded, then the lengths of the $\bar{\gamma}_i$ are all bounded by another constant depending only on Σ.*

This "a-priori bound" is proven in [Mi4], and one can see its relevance to the ELC given that tight geodesics are used to construct the model space. Minsky's argument is part of a larger project, and uses much sophisticated machinery. A more direct proof of this statement is given in [Bow5].

The vague idea is that, if the result should fail, we can find such a set-up in a 3-manifold in which at least some of the $\bar{\gamma}_i$ are very long. We can connect them by interlocking pleated surfaces, $\phi_i : (\Sigma, \rho_i) \longrightarrow M$. In these pleated surfaces, the very long γ_i will tend to "fill out" certain subsurfaces, $F_i \subseteq \Sigma$. Tightness means that γ_i must drag around with it either γ_{i-1} or γ_{i+1} (or both), so that F_i will have a homotopically non-trivial intersection with either F_{i-1} or F_{i+1}. We can then use this sequence of subsurfaces to shortcut the path $(\gamma_i)_i$, contradicting the assumption that it is geodesic in $\mathcal{G}(\Sigma)$.

To make proper sense out of this argument, we need at some point to use some kind of limiting procedure to derive a contradiction. As a result, we get some non-constructive input into the proceedings, and it is unclear whether the constant K featuring in Theorem 4.4 is a computable function of $g = \text{genus}(\Sigma)$. This therefore also applies to the constants in Theorem 2.4. Some algorithmic bounds associated tight geodesics are described in [Sha], showing for example that distances in $\mathcal{G}(\Sigma)$ are computable. However it seems more difficult to simultaneously achieve uniformity and computability of the various constants referred to earlier.

To conclude the proof of Theorem 4.4, one needs to delve further into the geometry of M. For this we use the band systems constructed in [Bow4]. A "band system" gives some kind of topological account of the failure of bounded geometry in M. One needs to argue that realisations of curves featuring in tight geodesics cannot enter any such band. The bounded geometry of M outside the bands then gives rise to combinatorial restrictions on the possibilities for such curves.

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