Planar groups and the Seifert conjecture

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0. Introduction.

In this paper, we describe a number of characterisations of virtual surface groups. One of the principal results will be:

Theorem 0.1 :   *Suppose $\mathbf{F}$ is a field, and that Γ is a group which is FP_2 over $\mathbf{F}$. If $H^2(\Gamma; \mathbf{F}\Gamma)$ has a 1-dimensional Γ-invariant subspace, then Γ is a virtual surface group.*

Recall that the FP_2 condition means that the trivial $\mathbf{F}\Gamma$-module, $\mathbf{F}$, admits a partial resolution $P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbf{F}$, where P_0, P_1 and P_2 are finitely generated projective $\mathbf{F}\Gamma$-modules. The cohomology group $H^2(\Gamma; \mathbf{F}\Gamma)$ has the structure of an $\mathbf{F}\Gamma$-module, or equivalently, a vector space over $\mathbf{F}$ with a linear Γ-action. We are therefore assuming that it has an $\mathbf{F}\Gamma$-submodule which is isomorphic to the trivial module. By a *virtual surface* group, we mean a group with a finite index subgroup which is isomorphic to the fundamental group of a closed surface other than the 2-sphere or projective plane. Note that it follows that, in fact, $H^2(\Gamma; \mathbf{F}\Gamma) \cong \mathbf{F}$, and that $H^n(\Gamma, \mathbf{F}\Gamma) = 0$ for all $n \neq 2$.

Theorem 0.1 applies in particular to 2-dimensional rational Poincaré duality groups:

Corollary 0.2 :   *A group Γ is $PD(2)$ over $\mathbf{Q}$ if and only if it is a virtual surface group.*

This answers affirmatively a conjecture of Dicks and Dunwoody (see [DiD] Chapter V, Conjecture 4.6).

For definitions and further discussion of Poincaré duality groups, see for example [Br,DiD]. The result of Eckmann, Müller and Linnell [EM,EL] characterises surface groups as $PD(2)$ groups over the integers. In view of the fact that torsion-free virtual surface groups are surface groups (following, for example, from [EM,EL]), Corollary 0.2 can be viewed as a generalisation of this result. In fact, Corollary 0.2 had already been established in the case where Γ is assumed to contain an infinite order element (see [DuSw]). In fact, much of the proof of Theorem 0.1 will be aimed at the elimination of the possibility that Γ might be a torsion group. Another approach to these results, which avoids this particular difficulty, has been given by Kleiner [Kl] (at least in the finitely presented case).

Another corollary of Theorem 0.1 is:

Corollary 0.3 :   *If Γ is finitely presented, one-ended, semistable at infinity, and $\pi_1^\infty(\Gamma) \cong \mathbf{Z}$, then Γ is a virtual surface group.*

The notion of "semistability at infinity" was defined in [Mi]. Here $\pi_1^\infty(\Gamma)$ denotes the fundamental group at infinity.

This, in turn, leads to another proof of the result originating in the work of Mess:

Corollary 0.4 : [T,Me,Ga,CJ] *If Γ is finitely generated and quasiisometric to a complete riemannian plane, then Γ is a virtual surface group.*

In fact, it's enough for Γ to be quasiisometric to any complete path-metric space homeomorphic to $\mathbf{R}^2$.

The logical interconnections between these various characterisations, along with several others, will be described in Section 14.

Our proof of Theorem 0.1 involves interpreting the homological condition in terms of a kind of "coarse planarity" of the Cayley graph of Γ. A number of different formulations might be given for this. We shall focus on one involving winding numbers, as described in Sections 1 and 2. This condition will easily be seen to be a quasiisometry invariant. Much of this material can be interpreted in terms of the theory of "coarse Alexander duality". This theory was introduced in [FarbS], and some related ideas can be found in the work of Higson and Roe (see [R]). It has been developed extensively in a more general context by Kapovich and Kleiner [KaK]. However, for the specific cases in which we deal here, it will be easy to give direct arguments.

One of the main intermediate goals in the proof will be to show that Γ has an infinite order element, and that every infinite cyclic subgroup is codimension-one. The argument can then be completed using a result from [Bo]. Graham Niblo has observed that this can also be deduced from the results and methods of [DuSa], [DuSw] and [Sw], at least in the case where Γ is almost finitely presented, and has suggested how the arguments might be adapted to deal with the finitely generated case. We shall also outline a more direct argument in Section 13. Both these approaches depend on the classification of convergence actions on the circle by Tukia, Gabai and Casson and Jungreis [T,Ga,CJ]. All the arguments involved are essentially geometric, and can, in principle be interpreted combinatorially.

In the original proof of Corollary 0.4, Mess makes use of Varopoulos's theorem (see [V] or [W]). This relies on Gromov's result [Gr] on groups of polynomial growth which in turn relies on the solution to Hilbert's fifth problem [MoZ]. Recently, Maillot [Ma] has given a more direct geometric proof of Corollary 0.4 which byepasses Varopoulos's result, though still requires that of Gromov. Kleiner's approach to Corollary 0.2 also depends on Gromov's result. However, this can be eliminated using results of the present paper (in particular Theorem 12.9), thereby giving another essentially combinatorial proof. All of these arguments rely on [T,Ga,CJ].

We note that conditions of the type appearing in Theorem 0.1 are discussed in the papers of Farrell [Farr1,Farr2]. For example, in [Farr2], he shows that if $\mathbf{F} = \mathbf{Z}_2$ and Γ is not torsion, then any finite dimensional Γ-invariant subspace of $H^2(\Gamma; \mathbf{F}\Gamma)$ is 1-dimension. It would be interesting to know if this can be extended to other fields, and whether the non-torsion hypothesis can be eliminated. If so, this would give a stronger version of Theorem 0.1.

In retrospect, we see from Theorem 0.1 that the "orientation preserving" subgroup of Γ, i.e. that fixing $H^2(\Gamma; \mathbf{F}\Gamma)$ pointwise, has index at most 2 in Γ. One can similarly define the orientation preserving subgroup of a Poincaré duality group in any dimension, n (replacing $H^2(\Gamma; \mathbf{F}\Gamma)$ by $H^n(\Gamma; \mathbf{F}\Gamma)$). It seems to be an open question as to whether this subgroup must always have index at most 2.

One of the main motivations of Mess's paper [Me] was to reduce the Seifert Conjecture to Corollary 0.4. The Seifert Conjecture states that if M is a closed irreducible 3-manifold such that $\pi_1(M)$ contains an infinite cyclic normal subgroup, then M is a Seifert fibred space. Theorem 0.1 allows us to give a version of this for $PD(3)$ groups:

Corollary 0.5 : *Suppose that Γ is $PD(3)$ over $\mathbf{Z}$, and contains an infinite cyclic normal subgroup. Then, Γ is the fundamental group of a closed Seifert fibred 3-manifold.*

This answers a question attributed to Scott in the problem list compiled by Kirby [Ki] (No. 3.77(B)). It was already known in the case where Γ is assumed to have infinite abelianisation [H1]. This result will be discussed further in Section 15. In view of the result of Scott [Sco1] that there are no "fake" Seifert fibre spaces with infinite fundamental group, we recover the Seifert conjecture as a corollary.

The main result of this paper also has applications to 4-manifolds. In particular, the fact that $H^2(\Gamma, \mathbf{Z}\Gamma) \cong \mathbf{Z}$ implies that Γ is a virtual surface group, for Γ FP_2 over $\mathbf{Z}$, can be used to streamline or strengthen a number of results in [H3] (for example, by eliminating hypotheses demanding the non-vanishing of first cohomology). Such applications are described in [H4].

I am indebted to several people, in particular, Michel Boileau, Warren Dicks, Martin Dunwoody and Jonathan Hillman for bringing these questions to my attention. I have profited particularly from discussion with Bruce Kleiner, who first suggested using some notion of "rotation number" in this context, and who explained to me the principles of coarse Alexander duality which have helped to streamline some of the arguments of this paper. I would also like to thank Ian Leary for his help with some of the more algebraic aspects of this paper. Discussions with John Crisp, Warren Dicks, Martin Dunwoody and Graham Niblo have also been helpful.

1. Notation and conventions.

Before starting properly, we describe some conventions and terminology used throughout this paper.

We shall use $\mathbf{A}$ and $\mathbf{F}$ to denote a commutative ring with a one and a field respectively. We write $\mathbf{F}^\times$ for the multiplicative group $\mathbf{F} \setminus \{0\}$. If Γ is a group, we write $\mathbf{A}\Gamma$ and $\mathbf{F}\Gamma$ for the corresponding group rings.

Suppose X is a locally finite 2-dimensional CW complex. We put a path-metric ρ on the 1-skeleton, X_1, by assigning unit length to every 1-cell. If there is a bound on the length of the boundary of each 2-cell, then we shall refer to (X, ρ) as a *metric 2-complex*. It is sometimes convenient to imagine ρ extended to all of X, in such a way that the 1-

skeleton is geodesically embedded, and such that there is a bound on the diameter of each 2-cell. However, only the metric on X_1 is strictly relevant.

We shall write $V(X)$ for the set of vertices of X. However, we shall take the statement $x \in X$ to mean implicitly that we are choosing a vertex of X. Similarly, we shall assume that all the subsets of X with which we are dealing are subcomplexes. Also paths in X are always assumed to map into the 1-skeleton. The only purpose in these conventions is to avoid having to worry about nasty subsets.

Suppose Q is a subset (i.e. subcomplex) of X. We write ρ_Q for the induced path-metric on (the 1-skeleton of) Q. We write Q^C for the closure of the complement of Q. Given $r \in \mathbf{N}$, we write $N(Q, r)$ for the subcomplex of X whose 1-skeleton, K, is the r-neighbourhood of the 1-skeleton of Q, and where a 2-cell of X lies in $N(Q, r)$ if and only if its boundary lies in K. We shall generally use this notation without bothering to specify that $r \in \mathbf{N}$.

We can think of a *path* in X formally as a cellular map of a subinterval of the real line into X_1. A *loop* is a closed path, and a *circuit* is an embedded loop. We shall speak of *finite paths*, *rays* and *biinfinite paths*, if the domain is compact, one-ended, or 2-ended respectively. We shall always assume rays and biinfinite paths to be proper maps. We shall frequently abuse notation, by identifying a path with its image in X, even if it is not embedded. A finite path is a *geodesic* if its length equals the distance between its endpoints. In general, a path is *geodesic* if every finite subpath is. If the direction of a path is important, we shall sometimes denote it by $\vec{\alpha}$ where α is the underlying undirected path. We shall write $-\vec{\alpha}$ for the same path directed in the opposite direction. We use $\cup$ for concatenation of paths.

We shall write $\mathcal{L}(X)$ for the set of loops in X. Note that there is natural map of $\mathcal{L}(X)$ into $H_1(X; \mathbf{A})$ for any ring $\mathbf{A}$. We shall write $\langle ., . \rangle$ for the Kronecker pairing on $H^1(X; \mathbf{A}) \times H_1(X; \mathbf{A}) \longrightarrow \mathbf{A}$. If $\mathbf{A} = \mathbf{F}$ is a field, then this is a non-degenerate bilinear form, so we can identify $H^1(X; \mathbf{F})$ as the dual space of $H_1(X; \mathbf{F})$.

We shall refer to a map $\mu : \mathcal{L}(X) \longrightarrow \mathbf{F}$ as a *cocycle* if it factors through a linear map of $H_1(X; \mathbf{F})$ to $\mathbf{F}$ (i.e. an element of $H^1(X; \mathbf{F})$). This linear map is uniquely determined, and we shall also denote it by μ. The cocycle condition can be expressed more combinatorially as follows. If r is a bound on the length of the boundary of any 2-cell, then $\mu : \mathcal{L}(X) \longrightarrow \mathbf{F}$ is a cocycle if and only if:

(1) $\mu(\gamma_1) + \mu(\gamma_2) + \mu(\gamma_3) = 0$ whenever $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{L}(X)$ form a theta-curve, and

(2) $\mu(\gamma) = 0$ whenever $\mathrm{length}(\gamma) \leq r$.

We say that $\gamma_1, \gamma_2, \gamma_3$ form a *theta curve* if there are finite paths $\alpha_1, \alpha_2, \alpha_3$ sharing the same pair of endpoints, such that $\gamma_i = \alpha_i \cup -\alpha_{i+1}$, taking subscrips mod 3. We shall speak of an *integral cocycle* to mean a map $\mu : \mathcal{L}(X) \longrightarrow \mathbf{Z}$ which extends to a cocycle with values in $\mathbf{Q}$. Such a map is also characterised by properties (1) and (2).

We shall say that X is $\mathbf{A}$-*acyclic* if $H_1(X; \mathbf{A}) = 0$. We say that X is *uniformly $\mathbf{A}$-acyclic* if for all $r \geq 0$ there exists $s \geq r$ such that for all $x \in X$, the image of $H_1(N(x, r); \mathbf{A})$ in $H_1(N(x, s); \mathbf{A})$ is 0. Note that these properties remain unchanged if we add additional 2-cells to X. We can thus speak of a locally finite graph as being stably (uniformly) acyclic if there is some $t \geq 0$ such that if X is the complex obtained by attaching a 2-cell to every loop in X of length at most t, then X is (uniformly) acyclic. We see that if a 2-complex is

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(uniformly) acyclic, then its 1-skeleton is stably uniformly acyclic. Moreover the property of stable (uniform) acyclicity is easily seen to be a quasiisometry invariant.

2. End cohomology and winding numbers.

In this section, we mention some basic facts regarding "end cohomology" and, in the 1-dimensional case, relate this to the more geometric notion of "winding number".

Let $\mathbf{A}$ be a ring, and let X be a locally finite n-dimensional CW complex, with $H^i(X; \mathbf{A}) = 0$ for all $i < n$. Suppose Γ acts freely cocompactly on X. Let $C_* = C_*(X; \mathbf{A})$ denote the cellular chain complex with coefficients in $\mathbf{A}$, thought of as a graded $\mathbf{A}\Gamma$-module. We get a finitely generated partial free resolution of $\mathbf{A}$:

$$C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathbf{A},$$

which we can extend arbitrarily to free resolution:

$$\cdots \longrightarrow C_{n+1} \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow \mathbf{A}.$$

Let $C^i = \text{Hom}(C_i; \mathbf{A})$ denote the cochain module, and let C_C^i denote the submodule of finitely supported cochains. Let $\delta : C^i \longrightarrow C^{i+1}$ be the coboundary map. Now, $\delta(C_C^i) \subseteq C_C^{i+1}$, so $(C_C^i, \delta)_i$ is a sub chain complex. We denote its homology by H_C^*. Now, since the cochain complex $(C^i)_i$ is exact, for any m, the submodule, Z^m of cochains in C^m is also the module of coboundaries $B^m = \delta(C^{m-1})$. Let $Z_C^m = C_C^m \cap Z^m$ and let $B_C^m = \delta(C_C^{m-1}) \subseteq Z_C^{m-1}$. By definition, $H_C^m(X; \mathbf{A}) = Z_C^m/B_C^m = (C_C^m \cap \delta(C^{m-1}))/\delta(C_C^{m-1})$. Now, for $m < n$, H_C^m is the "compactly supported cohomology" of X. Here we are interested in the group H_C^n which depends only on the complex X. We denote it by $J^n(X)$.

Now, if M is any $\mathbf{A}\Gamma$-module, then $\mathbf{A}\Gamma$-module homomorphisms from M into $\mathbf{A}$ can be identified with finitely supported $\mathbf{A}$-module homomorphisms from M into $\mathbf{A}$. This gives rise to a natural identification of $J^n(X)$ with $H^n(\Gamma; \mathbf{A}\Gamma)$. (See [Br] for details.)

We can also interpret this in terms of the "end cohomology" of X. Let $\mathcal{D}$ be the directed set of compact subcomplexes of X, ordered by inclusion. This gives a direct limit system of cohomology groups $(H^i(K^C; \mathbf{A}))_{K \in \mathcal{D}}$ (where K^C denotes the closure of the complement of K). We denote the direct limit by $H_\infty^i(X; \mathbf{A})$, and refer to it as the *end cohomology* of X. We can identify $J^n(X)$ with $H_\infty^{n-1}(X; \mathbf{A})$ as follows.

Suppose $\sigma \in J^n(X)$. Now, σ is represented by compactly supported coboundary, $\delta t \in C_C^n$ where $t \in C^{n-1}$. Choose some $K \in \mathcal{D}$ such that $\delta t \equiv 0$ on K^C. Thus, t defines an element of $H^{n-1}(K^C; \mathbf{A})$ which gives rise to an element, $f(\sigma)$, in the direct limit $H_\infty^{n-1}(X; \mathbf{A})$. It is easily verified that f is an abelian group isomorphism from $J^n(X)$ to $H_\infty^{n-1}(X; \mathbf{A})$. If $\mathbf{A} = \mathbf{F}$ is a field, then f is $\mathbf{F}$-linear.

Let us now restrict to the case where $n = 2$, and $\mathbf{F}$ is a field.

The following result was shown to me by Ian Leary. For more details, see [L].

Lemma 2.1 : *Let* **F** *be any field. Then* Γ *is* FP_2 *over* **F** *if and only if* Γ *acts properly discontinuously cocompactly on a locally finite 2-complex,* X, *with* $H_1(X; \mathbf{F}) = 0$.

Only the "only if" part is relevant here. (The "if" part is easy.) We begin by showing the result is true if **F** is replaced by the integers, **Z**. Note that FP_1 over **Z**, or indeed any non-trivial ring, is equivalent to finite generation [Br]. Thus, if Γ is FP_2 over **Z** we obtain an exact sequence of **Z**Γ-modules, $0 \longrightarrow K \longrightarrow C_1(Y; \mathbf{Z}) \longrightarrow C_0(Y; \mathbf{Z}) \longrightarrow \mathbf{Z} \longrightarrow 0$, where K is finitely generated, and $C_i(Y; \mathbf{Z})$ are the chain modules of some Cayley graph, Y, of Γ. Now, the image of K is $C_1(Y; \mathbf{Z})$ is generated, as an abelian group, by the set of Γ-images of a finite set of circuits in Y. By attaching 2-cells to circuits we obtain the desired 2-complex, X.

Now, the same argument works with **Z** replaced by $\mathbf{Z}_n$ for any n, or by **Q**. In the case of **Q** we need to note that some multiple of any element of $H_1(Y; \mathbf{Q})$ is a represented by a finite sum of circuits.

To complete the argument, we show that a group is FP_2 over a field, **F**, (if and) only if it is FP_2 over its prime subfield, **E**. To see this, consider the sequence $0 \longrightarrow K \longrightarrow C_1(Y, \mathbf{E}) \longrightarrow C_0(Y; \mathbf{E}) \longrightarrow \mathbf{E} \longrightarrow 0$ as above. This time, K is an **E**Γ-module such that $K \otimes_{\mathbf{E}\Gamma} \mathbf{F}\Gamma$ if is finitely generated as an **F**Γ-module. If $\{\sum_{j=1}^{n_i} k_{ij} \otimes \lambda_{ij} \mid i = 1, \ldots, m\}$ generates $K \otimes_{\mathbf{E}\Gamma} \mathbf{F}\Gamma$, where $k_{ij} \in K$ and $\lambda_{ij} \in \mathbf{F}$, then $\{k_{ij}\}$ generates K. To see this, let L be the submodule of K thus generated, so that we have an exact sequence $0 \longrightarrow L \longrightarrow K \longrightarrow M \longrightarrow 0$, where M is the cokernel. Since **F**Γ is faithfully flat as an **E**Γ-module, it follows that $M = 0$. We thus deduce that K is finitely generated as an **E**Γ-module as required. (The converse of the above statement is easy.)

In summary, if Γ is FP_2 over **F**, then it's FP_2 over the prime subfield, **E**. Thus Γ acts properly discontinuously cocompactly on a 2-complex, X, with $H_1(X; \mathbf{E}) = 0$. It follows by the Universal Coefficient Theorem that $H_1(X; \mathbf{F}) = 0$.

More elaboration of this will be given in [L]. We remark that not all finiteness properties of large fields pass to subfields. For example, there are examples of groups that are FL over **C** but not over **Q** — see [L].

Now, suppose that $E \subseteq H^2(\Gamma; \mathbf{F}\Gamma)$ is a Γ-invariant subspace. This gives us a 1-dimensional subspace of $H^1_\infty(X; \mathbf{F})$, which we also denote by E. We shall explain how this gives rise to a "winding number" in **F**. Firstly we should give some definitions.

Suppose (X, ρ) is a locally finite metric 2-complex, and that $r_0 \geq 0$ is some constant. We assume that the boundary of every 2-cell in X has length at most r_0. Suppose that $H_1(X; \mathbf{F}) = 0$. Given two subsets $P, Q \subseteq X$, we shall write $P \wedge Q$ to mean that $\rho(P, Q) \geq r_0$. Let $W = W(r_0, X) = \{(x, \beta) \in X \times \mathcal{L}(X) \mid x \wedge \gamma\}$. (We are tacitly assuming that x is vertex of X.)

Definition : A *winding number* on X with values in **F** (and with *separation constant* r_0) is a map $\omega : W \longrightarrow \mathbf{F}$ satisfying the following:
(W1) Given $x \in X$, the map $[\gamma \mapsto \omega(x, \gamma)] : \mathcal{L}(N(x, r_0)^C) \longrightarrow \mathbf{F}$ is a cocycle in $N(x, r_0)^C$ (i.e. factors through a linear map on homology),
(W2) if $x, y \in X$ are adjacent vertices (in the 1-skeleton of X), and $\gamma \in \mathcal{L}(X)$ with $x \wedge \gamma$ and $y \wedge \gamma$, then $\omega(x, \gamma) = \omega(y, \gamma)$, and

(W3) $(\exists x \in X)(\forall r \geq r_0)(\exists \gamma \in \mathcal{L}(X))\ \rho(x,\gamma) \geq r$ and $\omega(x,\gamma) \neq 0$.

Note that the above definition makes no reference to the group action. However, the winding number we construct will have an additional invariance property, namely:

(W4) There is a character θ such that for all that for all $g \in \Gamma$, $x \in X$ and $\gamma \in \mathcal{L}(X)$ with $x \wedge \gamma$, we have $\omega(gx, g\gamma) = \theta(g)\omega(x,\gamma)$.

(By a *character* we mean a homomorphism from Γ to $\mathbf{F}^{\times}$.)

We shall eventually see that it is necessarily the case that the image of θ lies in $\{-1, 1\}$, beyond which point we can pass to a subgroup of index at most 2 so that $\omega(gx, g\gamma) = \omega(x,\gamma)$ for all g.

We now describe how to construct such a winding number. Suppose Γ is FP_2 over $\mathbf{F}$, and X is an $\mathbf{F}$-acyclic 2-complex on which Γ acts freely and cocompactly. Note that X is necessarily uniformly $\mathbf{F}$-acyclic as defined in Section 1. Let $E \subseteq H^1_{\infty}(X; \mathbf{F})$ is an invariant 1-dimensional subspace. As an intermediate step, we have:

Lemma 2.2 : *There is a compact set $K \in \mathcal{D}$, a linear map $\mu : H_1(K^C; \mathbf{F}) \longrightarrow \mathbf{F}$, and a character $\theta : \Gamma \longrightarrow \mathbf{F}^{\times}$ such that for every $g \in \Gamma$, there is a compact set $L(g) \in \mathcal{D}$ with $K \cup g^{-1}K \subseteq L(g)$ such that if $\gamma, g\gamma \in \mathcal{L}(L(g))$, then $\mu(g\gamma) = \theta(g)\mu(\gamma)$. Moreover, if $L \in \mathcal{D}$ with $K \subseteq L$, then there is some $\gamma \in \mathcal{L}(L^C)$ with $\mu(\gamma) \neq 0$.*

(Here we are using the convention of using the same symbol to denote a linear map on $H_1(Q; \mathbf{F})$ and the induced map on $\mathcal{L}(Q)$ where $Q \subseteq X$.)

Proof : Choose any element $\phi \in E \setminus \{0\}$. We have a homomorphism, $\theta : \Gamma \longrightarrow \mathbf{F}^{\times}$, defined by $g.\phi = \theta(g^{-1})\phi$, where $[(g, \phi) \mapsto g.\phi]$ denotes the action of Γ on $E \subseteq H^1_{\infty}(X; \mathbf{F})$. Now, we can find a compact $K \in \mathcal{D}$ so that ϕ corresponds to some element $\psi \in H^1(K^C; \mathbf{F})$. Given any element $s \in H_1(K^C; \mathbf{F})$, we set $\mu(s) = \langle \psi, s \rangle$, where $\langle ., . \rangle$ denotes the Kronecker product.

Now, since $\phi \neq 0$, for any $L \in \mathcal{D}$ with $K \subseteq L$, we have $\psi | L^C \not\equiv 0$. Since $\mathcal{L}(L^C)$ spans $H_1(L^C; \mathbf{F})$, and the Kronecker product is non-degenerate, there is some $\gamma \in \mathcal{L}(L^C)$ with $\mu(\gamma) \neq 0$.

Suppose $g \in \Gamma$. Now, $g^{-1}.\phi = \theta(g)\phi$, so, by the definition of the Γ-action on $H^1_{\infty}(X; \mathbf{F})$, we have that $g^{-1}.\psi \in H^1(g^{-1}K^C; \mathbf{F})$ and $\theta(g)\psi \in H^1(K^C; \mathbf{F})$ pull back to the same element of $H^1(L(g); \mathbf{F})$ for some $L(g) \in \mathcal{D}$ with $K \cup g^{-1}K \subseteq L(g)$. Thus, if $\gamma, g\gamma \in \mathcal{L}(L(g)^C)$, then $\mu(g\gamma) = \langle \psi, g\gamma \rangle = \langle g^{-1}.\psi, \gamma \rangle = \langle \theta(g)\psi, \gamma \rangle = \theta(g)\langle \psi, \gamma \rangle = \theta(g)\mu(g)$ as required. $\diamondsuit$

We can now go on to construct a winding number having properties (W1)–(W4) as follows.

Fix an orbit transversal, A, of the set of vertices of X. Thus, A is finite. Let $S \subseteq \Gamma$ be a finite symmetric set such that if some element of A is adjacent to some vertex of gA, then $g \in S$. Choose $r_0 \geq 0$ big enough so that for any $x \in A$ and $g \in S$, then $K \subseteq N(x, r_0)$

and $L(g) \subseteq N(x, r_0)$. Thus, if $x \in A$, $\gamma \in \mathcal{L}(X)$ and $g \in S$ with $x \wedge \gamma$ and $x \wedge g\gamma$, then $\mu(g\gamma) = \theta(g)\mu(\gamma)$.

Now, given a vertex $x \in X$, there is a unique $g \in \Gamma$, such that $g^{-1}x \in A$. If $\gamma \in \mathcal{L}(X)$ with $x \wedge \gamma$, then $g^{-1}x \wedge g^{-1}\gamma$ and so $\mu(g^{-1}\gamma)$ is defined. We set $\omega(x, \gamma) = \theta(g)\mu(g^{-1}\gamma)$.

Note that if $y \in A$, then $\omega(y, \gamma) = \mu(\gamma)$ and $\omega(gy, g\gamma) = \theta(g)\mu(\gamma)$ for all $g \in \Gamma$. We deduce that for any vertex $x \in X$ with $x \wedge \gamma$ and any $g \in \Gamma$, $\omega(gx, g\gamma) = \theta(g)\omega(x, \gamma)$. In other words, property (W4) holds.

Suppose now that $x, y \in X$ are adjacent vertices, and that $x, y \wedge \gamma$. Let $g, h \in \Gamma$ be such that $x \in gA$ and $y \in ghA$. Now A and hA contain adjacent vertices (namely $g^{-1}x$ and $g^{-1}y$), so $h \in S$. Now, $g^{-1}x \wedge g^{-1}\gamma$ and $h^{-1}g^{-1}y \wedge h^{-1}g^{-1}\gamma$ and $g^{-1}x, h^{-1}g^{-1}x \in A$. Thus, $\mu(h^{-1}g^{-1}\gamma) = \theta(h^{-1})\mu(g^{-1}\gamma)$, and so $\omega(y, \gamma) = \theta(gh)\mu(h^{-1}g^{-1}\gamma) = \theta(gh)\theta(h^{-1})\mu(g^{-1}\gamma) = \theta(g)\mu(g^{-1}\gamma) = \omega(x, \gamma)$. This proves property (W3).

Finally we note that properties (W1) and (W2) are immediate, so we have constructed our winding number as claimed.

Definition : We say that a locally finite metric 2-complex, (X, ρ), is *homologically planar* over a field $\mathbf{F}$, if it is uniformly $\mathbf{F}$-acyclic over $\mathbf{F}$, and there is some $r_0 \geq 0$ such that the boundary of every 2-cell has length at most r_0, and such that X admits a winding number satisfying axioms (W1), (W2) and (W3).

Note that this property remains invariant under attaching additional 2-cells of bounded boundary length. It can thus be viewed as a property of graphs, and as such is easily seen to be a quasiisometry invariant. It thus makes sense to speak about a finitely generated group as being "planar" in the sense that some (hence every) Cayley graph has this property. Note that planarity implies FP_2 over $\mathbf{F}$. We shall eventually see that it implies that Γ is a virtual surface group.

We can summarise the construction of this section in the following way:

Proposition 2.3 : *Suppose that Γ is a group and $\mathbf{F}$ is a field. Suppose Γ is FP_2 over $\mathbf{F}$ and that $H^2(\Gamma; \mathbf{F}\Gamma)$ has a 1-dimensional Γ-invariant subspace. Then Γ admits a free cocompact action on a metric 2-complex which is homologically planar over $\mathbf{F}$.* $\diamond$

Indeed the construction gave us axiom (W4) as well for free. We shall see in Section 6 how one can recover (W4) by purely geometric arguments, starting with assumption of planarity.

3. Uniform acyclicity and straight sets.

The kinds of ideas we describe here are related to "coarse Alexander duality". Similar ideas feature in [FarbS] and have been developed extensively in [KaK]. However, we shall only be using fairly simple properties which are easily derived from first principles.

Let $\mathbf{F}$ be a field and (X, ρ) be a metric 2-complex. Suppose $Q \subseteq X$, and $r \geq 0$.

Definition : We say that Q is *r-straight* (over $\mathbf{F}$) if the image of $H_1(Q; \mathbf{F})$ in $H_1(N(Q, r); \mathbf{F})$ is 0.

We say that Q is *straight* if it is r-straight for some r.

Thus, we can define uniformly acyclicity by saying that X is uniformly acyclic over $\mathbf{F}$ if there is some function $h : [0, \infty) \longrightarrow [0, \infty)$, such that if $Q \subseteq X$ with $\mathrm{diam}(Q) \leq r$, then Q is $h(r)$-straight.

We note that it is enough to verify this for circuits in X. Indeed, if the complex is uniformly locally finite, then it's enough that any circuit γ should bound an $\mathbf{F}$-cycle in X whose diameter is controlled as a function of length(γ).

Here are a couple of trivial observations about straightness.

Lemma 3.1 : *Suppose $Q \subseteq P \subseteq X$ are such that the natural map $H_1(Q; \mathbf{F}) \longrightarrow H_1(P; \mathbf{F})$ is surjective. If Q is r-straight, then so is P.* $\diamond$

Given $Q \subseteq X$, let $\mathcal{C}(Q)$ be the set of connected components of Q^C. As an immediate corollary of Lemma 3.1, we have

Lemma 3.2 : *Suppose X is acyclic, $Q \subseteq X$ and $\mathcal{E} \subseteq \mathcal{C}(Q)$. If Q is r-straight, then so is $Q \cup \bigcup \mathcal{E}$.* $\diamond$

Examples of straight subsets arise from the following construction. Suppose (Y, σ) and (X, ρ) are metric complexes. Suppose $f : (Y, \sigma) \longrightarrow (X, \rho)$ is a cellular map.

Definition : We say that f is a *uniform* map if for all $t \geq 0$ there is some $s \geq 0$ such that if $x, y \in Y$ with $\sigma(x, y) \geq s$, then $\rho(f(x), f(y)) \geq t$.

The following is easily verified (cf. [KaK]):

Lemma 3.3 : *Suppose (Y, σ) and (X, ρ) are both uniformly acyclic, and $f : Y \longrightarrow X$ is a uniform map. Then, for all $r \geq 0$, $N(f(Y), r)$ is s-straight, where s depends only on r and the functions of uniformity.* $\diamond$

We now want to apply these ideas to planar complexes. Recall that a "planar complex" is a uniformly $\mathbf{F}$-acyclic metric 2-complex, (X, ρ), with a winding number ω satisfying axioms (W1)–(W3).

Suppose $Q \subseteq X$ is connected. If $x, y \in Q$ and $\gamma \in \mathcal{L}(X)$ with $Q \wedge \gamma$, then axiom (W2) tells us that $\omega(x, \gamma) = \omega(y, \gamma)$. We shall denote this quantity by $\omega(Q, \gamma)$.

Lemma 3.4 : *For all $t \geq 0$, there is some $r \geq r_0$ such that if $x \in X$ and $\gamma \in \mathcal{L}(X)$ with $\rho(x, \gamma) \geq r$ and $\mathrm{diam}(\gamma) \leq t$, then $\omega(x, \gamma) = 0$.*

Proof : By uniform acyclicity of X, there is some $s \geq 0$ depending on t such that γ is $\mathbf{F}$-homologous to 0 in $N(\gamma, s)$. Let $r = r_0 + s$. If $\rho(x, \gamma) \geq r$, then $N(\gamma, s) \cap N(x, r_0) = \emptyset$. Thus, γ is null $\mathbf{F}$-homologous in $N(x, r_0)^C$, so $\omega(x, \gamma) = 0$. $\diamond$

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Corollary 3.5 : *For all $t \geq 0$ there is some $u \geq 0$ such that if $Q \subseteq X$ is connected and $\gamma \in \mathcal{L}(X)$ with $Q \wedge \gamma$, $\operatorname{diam}(Q) \geq u$ and $\operatorname{diam}(\gamma) \leq t$, then $\omega(Q, \gamma) = 0$.*

Proof : Let $u = t + 2r$, where r is given by Lemma 3.4. Now, we can find $x \in Q$ with $\rho(x, \gamma) \geq r$. Thus, $\omega(Q, \gamma) = \omega(x, \gamma) = 0$. $\diamond$

As an immediate consequence, we have:

Lemma 3.6 : *Suppose $Q \subseteq X$ is connected and unbounded, and that $\gamma \in \mathcal{L}(X)$ with $Q \wedge \gamma$. Then $\omega(Q, \gamma) = 0$.* $\diamond$

Definition : A sequence $(\gamma_n)_{n \in \mathbf{N}}$ of loops in X is *big* if for some (hence every) $x \in X$, $\rho(x, \gamma_n) \to \infty$ and $\omega(x, \gamma_n) \neq 0$ for all sufficiently large n.
A subset $Q \subseteq X$ is *big* if it contains a big sequence of loops.

Thus, Axiom (W3) tells us that X itself is big.

Lemma 3.7 : *If $Q \subseteq X$ is big and r-straight, then $X = N(Q, r + r_0)$.*

Proof : Suppose $x \in X$ with $\rho(x, Q) > r + r_0$. If $\gamma \in \mathcal{L}(Q)$, then γ is null $\mathbf{F}$-homologous in $N(Q, r) \subseteq N(x, r_0)^C$. Thus $\omega(x, \gamma) = 0$ contradicting bigness. $\diamond$

As a consequence, one can show:

Proposition 3.8 : *A planar metric 2-complex is one-ended.*

Proof : Suppose $K \subseteq X$ is compact. Since X is locally finite, $\mathcal{C}(K)$ is finite. Let $(\gamma_n)_n$ be a big sequence in X. Passing to a subsequence, we can assume that each γ_n lies in C for some $C \in \mathcal{C}(K)$. Thus $C \cup K$ is big. Since X is uniformly acyclic, K is r-straight for some $r \geq 0$. Thus, by Lemma 3.2, $C \cup K$ is also r-straight. By Lemma 3.7, $X = N(C \cup K, r + r_0)$. It follows that K is the only unbounded element of $\mathcal{C}(K)$. This shows that X is one-ended. $\diamond$

4. Systems of connected sets.

For the moment, we shall allow (X, ρ) to be any metric 2-complex. Let $\mathcal{A}$ be a collection of connected subsets of X. By the *nerve*, $\Omega = \Omega(\mathcal{A})$, of $\mathcal{A}$, we mean the graph with vertex set $V(\mathcal{A}) \equiv \mathcal{A}$, and with two such vertices connected by an edge in Ω if and only if the corresponding sets have non-empty intersection. We shall adopt the convention of denoting elements of $\mathcal{A}$ by upper case letters, $A, B, C, \ldots$, and the corresponding vertices of Ω by the corresponding lower case letters, $a, b, c, \ldots$. We shall denote a path in Ω by listing the vertices through which it passes.

Given a loop, $\beta \in \mathcal{L}(\bigcup \mathcal{A})$, we shall say that a path $a_1 a_2 \ldots a_n$ is a *coding* for β if we can write $\beta = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$, with $\alpha_i \subseteq A_i \cup A_{i+1}$ for all i (subscripts mod n). We shall frequently write $\bar{\beta}$ for such a coding (even though it need not be uniquely determined).

By a *p-cycle* we mean a system $\mathcal{A}$ of connected sets such that $\Omega(\mathcal{A})$ is a p-gon (a circuit of length p), which we typically denote by $a_1 a_2 \ldots a_p$, so that $\mathcal{A} = \{A_1, \ldots, A_p\}$, taking subscripts mod p.

Lemma 4.1 : *Suppose $\mathcal{A} = \{A_1, \cdots, A_p\}$ is a p-cycle with $p \geq 4$, and $\beta \in \mathcal{L}(\bigcup \mathcal{A})$ has coding $a_1 a_2 \ldots a_p$. Then β represents a non-zero element of $H_1(\bigcup \mathcal{A}; \mathbf{A})$ for any ring $\mathbf{A}$.*

Proof : This follows from the following observation. Suppose C and D are connected complexes, and $x, y \in C \cap D$ lie in different components of $C \cap D$. If $\gamma \subseteq C$ and $\delta \subseteq D$ are paths each with endpoints x and y, then $\gamma \cup \delta$ is non-trivial in $H_1(C \cup D; \mathbf{A})$.

We now apply this to $C = A_1 \cup A_2 \cup A_3$ and $D = A_3 \cup A_4 \cup \cdots \cup A_p \cup A_1$. Then $C \cap D = A_1 \sqcup A_3$. Let $\gamma = \alpha_1 \cup \alpha_2$ and $\delta = \alpha_3 \cup \cdots \cup \alpha_p$ where $\alpha_i \subseteq A_i \cup A_{i+1}$. Thus, $\beta = \gamma \cup \delta$. $\diamond$

Now, fix a field, $\mathbf{F}$, and recall the definition of "straightness" from Section 3.

Lemma 4.2 : *Suppose that $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ is a 4-cycle in X, and that $\bigcup \mathcal{A}$ is r-straight in X. Then, either $\rho(A_1, A_3) \leq 2r$, or $\rho(A_2, A_4) \leq 2r$.*

Proof : Suppose not. Let $B_i = N(A_i, r)$. Then $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$ is a 4-cycle. Choose any loop $\beta \in \mathcal{L}(\bigcup \mathcal{A})$ with coding sequence $a_1 a_2 a_3 a_4$ in $\Omega(\mathcal{A})$ and hence $b_1 b_2 b_3 b_4$ in $\Omega(\mathcal{B})$. By straightness, β is trivial in $H_1(\bigcup \mathcal{B}; \mathbf{F})$, contrary to Lemma 3.1. $\diamond$

Lemma 3.3 : *Suppose $\mathcal{A}$ is a system of connected sets. Suppose that $I \leq H_1(\bigcup \mathcal{A}; \mathbf{F})$ is an $\mathbf{F}$-subspace with the property that for $A, B \in \mathcal{A}$, the image of $H_1(A \cup B; \mathbf{F})$ in $H_1(\bigcup \mathcal{A}; \mathbf{F})$ lies in I. Then, there is a natural map $j : H_1(\Omega; \mathbf{F}) \longrightarrow H_1(\bigcup \mathcal{A}; \mathbf{F})/I$ such that if $\beta \in \mathcal{L}(\bigcup \mathcal{A})$ with coding sequence $\bar{\beta}$ then the homology class of $\bar{\beta}$ maps to the representative of the homology class of β.* $\diamond$

Proof : If $\pi = a_1 \ldots a_n$ is a loop in Ω then choose $\beta \in \mathcal{L}(\bigcup \mathcal{A})$ with coding π, and set $j(\pi)$ to be the class of β. If β' were another such path, then it is easily seen that the homology class of $\beta - \beta'$ can be represented by $\sum_{i=1}^{n} \gamma_i$, where $\gamma_i \in \mathcal{L}(A_i \cup A_{i+1})$, and thus lies in I. This shows that $j(\pi)$ depends only on the loop π.

Now, if π is homologically trivial, then it's homotopically trivial, and so can be reduced to the trivial loop by a series of reductions of the form $aba \to a$, where aba is a subpath. Now, shortcutting β by cutting away a loop in $A \cup B$, we similarly reduce β to the trivial loop, and so β represents an element of I.

We now extend j $\mathbf{F}$-linearly over $H_1(\Omega; \mathbf{F})$. $\diamond$

Lemma 4.4 : *Suppose $\mathbf{F}$ is a field. Suppose $\mu \in H^1(\bigcup \mathcal{A}; \mathbf{F})$ and that the restriction of μ to $A \cup B$ is trivial for all $A, B \in \mathcal{A}$. Then, there is a (unique) $\bar{\mu} \in H^1(\Omega; \mathbf{F})$ such that $\beta \in \mathcal{L}(X)$ and $\bar{\beta}$ is a coding for β, then $\bar{\mu}(\bar{\beta}) = \mu(\beta)$.*

Proof : Let $I \leq H_1(\bigcup \mathcal{A}; \mathbf{F})$ be the subspace mapped to 0 by μ under the Kronecker pairing. Let j be as given by Lemma 3.3. Given $\zeta \in H_1(\Omega; \mathbf{F})$ set $\bar{\mu}(\zeta) = \mu(j(\zeta))$. This defines an element of $H^1(\Omega; \mathbf{F})$. $\diamondsuit$

5. Planar separation by paths.

Any properly embedded arc in the plane, $\mathbf{R}^2$, separates it into two components. In this section, we shall describe an analogous course separation property for uniform bi-infinite arcs in planar 2-complexes. In the setting of riemannian geometry, a similar coarse separation property can be found in [Sch] (see also [FarbS]).

We shall fix a field, $\mathbf{F}$, and use the term *prime subring* for the ring of integers in the prime subfield. This is $\mathbf{Z}$ if $char(\mathbf{F}) = 0$ and $\mathbf{Z}_p$ if $char(\mathbf{F}) = p$.

Let (X, ρ) be a metric 2-complex. A *uniform path* is a map $\alpha : I \longrightarrow X$, where I is a subinterval of the real line, satisfying $f(|t - u|) \leq \rho(\alpha(t), \alpha(u)) \leq |t - u|$, where $f : [0, \infty) \longrightarrow [0, \infty)$ is a fixed function tending to infinity (the "function of uniformity"). In this section, we won't worry much about parametrisation, but keep track of the direction of the path. We shall abuse notation by writing α for the image of α. Note that the notion of a uniform path can be defined without explicit reference to the parametrisation by saying that the diameter of any finite subarc is bounded above by some function of the distance between its endpoints. A uniform bi-infinite path is necessarily proper.

Given $x, y \in \alpha$, we write $\alpha[x, y]$ for the subarc $\alpha([t, u])$, where $x = \alpha(t)$ and $y = \alpha(u)$. Given a finite subarc, $\delta \subseteq \alpha$, we write $\alpha^-(\delta) = \alpha((-\infty, t])$ and $\alpha^+(\delta) = \alpha([u, \infty))$ where $\delta = \alpha([t, u])$. We use the same notation, $\alpha^\pm(x)$ if $x = \delta$ is just a point of α.

The following is a simple consequence of uniformity:

Lemma 5.1 : *Suppose that α is a uniform bi-infinite path, $x \in \alpha$ and $r \geq 0$. There there is a finite subpath $\delta \subseteq \alpha$ such that $\rho(x, \alpha^-) \geq r$, $\rho(x, \alpha^+) \geq r$, $\rho(\alpha^-, \alpha^+) \geq r$ and $\delta \subseteq N(x, l)$, where l depends only on r and the functions of uniformity.* $\diamondsuit$

The following is immediate from Lemma 4.3:

Lemma 5.2 : *If X is uniformly acyclic, and α is a uniform arc, then $N(\alpha, r)$ is s-straight, where s depends only on r and the functions of uniformity.* $\diamondsuit$

Now, let's suppose that (X, ρ) is a planar metric 2-complex, with winding number ω with separation constant r_0. Let's fix a uniform bi-infinite arc α. Let $r \geq r_0$, and set $Y = N(\alpha, r)$. Note that Y is topologically two-ended. Let $\mathcal{C} = \mathcal{C}(Y)$ be the set of components of Y^C.

Suppose $\delta \subseteq \alpha$. Write $\alpha^\pm = \alpha^\pm(\delta)$. Set $K = K(\delta) = N(\delta, r)$ and set $A^\pm = A^\pm(\delta) = N(\alpha^\pm, r)$. We shall want to assume that $\rho(A^+, A^-) > 0$ is sufficiently large depending on

Lemmas 5.2 and 4.2 as we shall explain in due course.

We note that the following two observations are both simple consequences of the uniformity of α.

Lemma 5.3 : *For all $t \geq 0$ there is some $u \geq 0$ such that if $x, y \in A^{\pm}$ with $\rho(\delta, x) \geq u$ and $\rho(\delta, y) \geq u$, then x and y are connected by a path $\epsilon \subseteq A^{\pm}$ with $\rho(\delta, \epsilon) \geq t$.* $\diamondsuit$

Lemma 5.4 : *For all $t \geq 0$ there is some $u \geq 0$ such that if γ is a path connecting A^+ to A^- in $N(Y, t)$, then $\rho(x, \gamma) \leq u$ for all $x \in \delta$.* $\diamondsuit$

In both cases, u depends only on t and the functions of uniformity.

Now, let $\mathcal{C}_0 = \{C \in \mathcal{C} \mid C \cap A^- \neq \emptyset \text{ and } C \cap A^+ \neq \emptyset\}$. Let $\mathcal{B} = \mathcal{C} \cup \{A^-, A^+\}$ and let $\mathcal{B}_0 = \mathcal{C}_0 \cup \{A^-, A^+\}$. Let $\Omega = \Omega(\mathcal{B})$ and let Ω_0 be the subgraph $\Omega(\mathcal{B})$ (as defined in Section 4). We recall the convention of using upper and lower case letters for corresponding elements of $\mathcal{B}$ and Ω.

Now, Ω_0 is the complete bipartite graph on the sets $\{a^-, a^+\}$ and $\{c \mid C \in \mathcal{C}_0\}$. The remainder of Ω consists of a number (possibly 0) of free edges attached to either a^- or a^+, and a (finite) number of isolated vertices.

Now, by Lemma 5.2, Y is s-straight for some fixed $s \geq 0$. Moreover, by Lemma 3.2, if $\mathcal{E} \subseteq \mathcal{C}$, then $Y \cup \bigcup \mathcal{E}$ is s-straight.

We shall now assume that $\rho(A^-, A^+) > 2s$.

Lemma 5.5 : *If $C \in \mathcal{C}_0$, then $\rho(K, C) \leq 2s$.*

Proof : Suppose $\rho(K, C) > 2s$. Then, $\{K, A^-, C, A^+\}$ is a 4-cycle in X. Moreover, $K \cup A^- \cup C \cup A^+ = Y \cup C$ is s-straight. Since $\rho(A^-, A^+) > 2s$, this contradicts Lemma 4.2. $\diamondsuit$

By the local finiteness of X, it follows that $\mathcal{C}_0$ is finite. Hence, Ω_0 is finite.

Now, let μ be the cocycle on $\bigcup \mathcal{B}$ defined by $[\gamma \mapsto \omega(\delta, \gamma)]$.

Lemma 5.6 : *If $B, C \in \mathcal{B}$ and $\gamma \in \mathcal{L}(B \cup C)$, then $\mu(\gamma) = 0$.*

Proof : We can assume that $B \cap C \neq \emptyset$. Thus, without loss of generality, $B, C \neq A^+$, and so $(B \cup C) \wedge (\delta \cup \alpha^+)$. But $\delta \cup \alpha^+$ is connected and unbounded. Thus, by Lemma 3.6, $\mu(\gamma) = \omega(\delta, \gamma) = 0$ for all $\gamma \subseteq B \cup C$. $\diamondsuit$

Thus, by Lemma 4.4, we get an element $\bar{\mu} \in H^1(\Omega; \mathbf{F})$, such that if $\beta \in \mathcal{L}(\bigcup \mathcal{B})$ with coding $\bar{\beta}$, then $\mu(\beta) = \bar{\mu}(\bar{\beta})$.

Now suppose $\beta \in \mathcal{L}(\bigcup \mathcal{B})$ with $\mu(\beta) \neq 0$. Let $\bar{\beta}$ be a coding for β. Now, from the form of Ω described above, we see that there must be subpaths of $\bar{\beta}$ of the form $a^- c_1 a^+$ and $a^+ c_2 a^-$, where $C_1, C_2 \in \mathcal{C}_0$ are distinct, and with $\bar{\mu}(a^- c_1 a^+ c_2) \neq 0$. It follows that β contains subarcs $\beta_1 \subseteq C_1$ connecting A^- to A^+ and $\beta_2 \subseteq C_2$ connecting A^+ to A^-. Let $\epsilon^{\pm}$ be a path in $A^{\pm}$ connecting the corresponding endpoints of β_1 and β_2, and set $\beta' = \epsilon^- \cup \beta_1 \cup \epsilon^+ \cup \beta_2 \in \mathcal{L}(\bigcup \mathcal{B})$. Now, β' has coding sequence $a^- c_1 a^+ c_2$, and so $\mu(\beta') \neq 0$.

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Note that, using Lemma 3.5, we can arrange that if $\rho(\delta, \beta) \geq u$, then $\rho(\delta, \beta') \geq t$, where u depends only on t.

Now, let $(\beta_n)_n$ be a big sequence in X. We can assume that $\beta_n \in \mathcal{L}(\bigcup \mathcal{B})$ for all n, and so we get a sequence β'_n constructed as above. Clearly, (β'_n) is also big. Moreover, since $\mathcal{C}_0$ is finite, passing to a subsequence, we can assume that β'_n has coding $a^- c_1 a^+ c_2$ for fixed $C_1, C_2 \in \mathcal{C}_0$ with $C_1 \neq C_2$. It follows that $Y \cup C_1 \cup C_2 \supseteq A^- \cup C_1 \cup A^+ \cup C_2$ is big.

Now, Y is s-straight, and so, by Lemma 3.3, $Y \cup C_1 \cup C_2$ is s-straight. Thus, by Lemma 3.7, we have $X = N(Y \cup C_1 \cup C_2, r_0 + s)$. Now, if $C \in \mathcal{C} \setminus \{C_1, C_2\}$, the nearest point in $Y \cup C_1 \cup C_2$ to any point in C must lie in Y. It follows that $C \subseteq N(Y, r_0 + s)$.

Suppose that $C_1 \subseteq N(Y, t)$ for some $t \geq 0$. Lemma 5.4 tells us that any path from A^- to A^+ in C_1 must lie a bounded distance from δ. But this contradicts the existence of our big sequence $(\beta'_n)_n$. We conclude that $C_1 \setminus N(Y, t) \neq \emptyset$ for all $t \geq 0$. The same goes for C_2.

In summary, we have shown:

Proposition 5.7 : *Suppose (X, ρ) is a planar metric 2-complex, $\alpha \subseteq X$ is a uniform bi-infinite path, and $r \geq r_0$. Let $Y = N(\alpha, r)$. There is a constant $k \geq r$, depending only on r and the functions of uniformity, and distinct elements $C_1, C_2 \in \mathcal{C}(Y)$ such that $C_1 \setminus N(\alpha, t) \neq \emptyset$ and $C_2 \setminus N(\alpha, t) \neq \emptyset$ for all $t \geq 0$ and $C \subseteq N(\alpha, k)$ for all $C \in \mathcal{C} \setminus \{C_1, C_2\}$.*
$\diamondsuit$

In particular, C_1 and C_2 are canonically determined by Y. We shall refer to them as the *deep* complements of Y. The remaining elements of $\mathcal{C}$ are *shallow*.

Now, by Lemma 5.4, we may as well choose k so that any path from A^- to A^+ must intersect $N(x, k)$ for any $x \in \delta$. Suppose that $\beta \in \mathcal{L}(\bigcup \mathcal{B})$ has coding $\bar{\beta}$, and that $\rho(\delta, \beta) > k$. Now if $\bar{\beta}$ contains a subpath $a^- c a^+$ or $a^+ c a^-$, then if follows that $c \in \{c_1, c_2\}$. Thus, by the description of Ω given earlier, we see that $\bar{\beta}$ is homotopic in Ω to some integral multiple, $\deg(\bar{\beta})$, of $a^- c_1 a^+ c_2$. In fact, the quantity $\deg(\bar{\beta})$ can be read off combinatorially from β as follows.

Let $d_i^\pm$ be the number of subpaths, γ, of β with the property that $\gamma \subseteq C_1$ and with γ meeting $A^\mp$ precisely in its initial point, and meeting $A^\pm$ precisely in its final point. Then it's easy to check that $\deg(\bar{\beta}) = d_1^+(\beta) - d_1^-(\beta) = d_2^-(\beta) - d_2^+(\beta) \in \mathbf{Z}$. We denote this quantity by $\deg(\beta) = \deg_{\alpha, \delta}(\beta)$.

Now, writing $\omega_0(\alpha, \delta) = \bar{\mu}(a^- c_1 a^+ c_2)$, we see that if $\beta \in \mathcal{L}(X)$ with $\rho(\delta, \beta) \geq k$, then $\omega(\delta, \beta) = \mu(\beta) = \deg(\beta) \omega_0(\alpha, \delta)$. In other words, $\omega(\delta, \beta)$ can be read off combinatorially.

We next show that $\omega_0(\alpha, \delta)$ is, in fact, independent of δ. Suppose that δ' is another such subarc of α. Since δ and δ' are contained in a common subarc, we may as well suppose that $\delta \subseteq \delta'$. Thus, $A^\pm(\delta') \subseteq A^\pm(\delta)$. Choose $\beta \in \mathcal{L}(X)$ with $\rho(\delta', \beta) > k$, and with $\deg_{\alpha, \delta'}(\beta) = 1$. Since $\beta \cap Y \subseteq A^-(\delta') \cup A^+(\delta')$, we see that $\deg_{\alpha, \delta}(\beta) = \deg_{\alpha, \delta'}(\beta) = 1$. Thus, $\omega_0(\alpha, \delta) = \omega(\delta, \beta) = \omega(\delta', \beta) = \omega_0(\alpha, \delta')$. We can therefore write $\omega_0(\alpha, \delta) = \omega_0(\alpha)$.

In summary, we see that $\omega_0(\alpha)$ depends only on the direction of the path α, and on an ordering on the pair of deep complements. We shall refer to the latter as an *orientation* on α. Given such an orientation, we shall write $C_R = C_1$ and $C_L = C_2$ for the *right* and *left* deep components respectively. Note that we can find a big sequence, $(\beta_n)_n$ such that

$\omega(\delta, \beta_n) = \omega_0(\alpha)$ for all n.

Next, we consider what happens if we choose a different uniform path α' with r' and k' corresponding constants. Choose a sufficiently large finite subarc, $\delta' \subseteq \alpha'$. Let $(\beta_n)_n$ be a big sequence with $\omega(\delta, \beta_n) = \omega_0(\alpha)$ for all n. Now, for all sufficiently large n, we have $\omega(\delta', \beta_n) = \omega(\delta, \beta_n)$ and that $\omega(\delta', \beta_n)$ is an integral multiple of $\omega_0(\alpha')$. In other words, $\omega_0(\alpha)$ is an integral multiple of $\omega_0(\alpha')$. Conversely, swapping the roles of α and α', we see that $\omega_0(\alpha')$ is an integral multiple of $\omega_0(\alpha)$. We thus conclude that $\omega_0(\alpha') = \lambda\omega_0(\alpha)$ where λ is a unit in the prime subring of $\mathbf{F}$, i.e. $\lambda = \pm 1$ if the characteristic of $\mathbf{F}$ is 0, and λ is a non-zero element of the prime subfield in general.

In summary, we have shown:

Proposition 5.8 :   *Suppose (X, ρ) is a planar metric 2-complex over $\mathbf{F}$. There, there is a non-zero element, $\omega_0 \in \mathbf{F}$, with the following property. Suppose $\alpha \subseteq X$ is a directed oriented uniform bi-infinite path. Then, there is some $\lambda \in \mathbf{F}$, which is a unit in the prime subring, satisfying the following. Suppose that $r \geq r_0$, then there is a constant $k \geq r$, depending only on r and the function of uniformity of α, such that if $\delta \subseteq \alpha$ is a sufficiently large finite subarc, and $\beta \in \mathcal{L}(X)$ with $\rho(\delta, \beta) \geq k$, then $\omega(\delta, \beta) = \deg(\beta)\lambda\omega_0$, where $\deg(\beta) \in \mathbf{Z}$ is the combinatorial degree of β as defined earlier.* $\diamond$

Of course, we haven't yet said anything about the existence of uniform bi-infinite paths, so the above result may be vacuous for all we know. If some bi-infinite uniform path exists, then the constant, ω_0, is determined up to a unit in the prime subring.

Definition : A metric 2-complex, (X, ρ), is *taut*, if there is some function $f : [0, \infty) \longrightarrow [0, \infty)$ tending to infinity, and some $r \geq 0$, such that every point of X lies within a distance r of some f-uniform bi-infinite path.

Note that by incorporating r into f, we may as well assume that every point of X lies on an f-uniform bi-infinite path.

Now, suppose that (X, ρ) is a taut planar metric 2-complex. Given $x \in X$, we can find a uniform α containing x. By Lemma 5.3, we can find a finite subpath $\delta \subseteq \alpha$, such that $\rho(\alpha^-(\delta), \alpha^+(\delta))$ is sufficiently large for the above constructions to work, and with $\delta \subseteq N(x, t)$, where t depends only on the parameters of planarity and tautness. Putting this together with Proposition 5.8, we conclude:

Proposition 5.9 :   *Suppose (X, ρ) is a taut planar 2-complex. Then there is some $l \geq r_0$, depending only on the parameters of planarity and tautness, and some non-zero $\omega_0 \in \mathbf{F}$, such that if $x \in X$ and $\gamma \in \mathcal{L}(X)$ with $\rho(x, \gamma) \geq l$, then $\omega(x, \gamma)$ is an integral multiple of ω_0.* $\diamond$

Thus, by enlarging r_0 to l, we may as well assume that the image of the winding number is precisely $\mathbf{Z}\omega_0$. Dividing throughout by ω_0, we can also assume that $\omega_0 = 1$. Thus, we will not loose any generality in assuming that either $\mathbf{F} = \mathbf{Q}$, and that the winding number takes precisely integral values, or that $\mathbf{F} = \mathbf{Z}_p$ for some prime p. In the latter case, we shall show in Section 7, how one can "lift" the winding number to $\mathbf{Z}$. Thus, ultimately, we will be able to assume that all winding numbers take values in $\mathbf{Z}$.

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The following definition will be useful in later sections to avoid having to deal with shallow components.

Suppose α is a uniform bi-infinite path, and $r \geq r_0$. Let $k = k(r) \geq r$ be the constant described by Proposition 5.7. Let $Y = N(\alpha, r)$ and let $\Lambda = \Lambda(\alpha, r) = Y \cup \bigcup(\mathcal{C}(Y) \setminus \{C_L, C_R\})$. Thus, $N(\alpha, r) \subseteq \Lambda \subseteq N(\alpha, k(r))$. Clearly, Λ has precisely two ends.

Suppose $\delta \subseteq \alpha$ is a finite subarc. Now $N(\alpha^+(\delta), k(r))$ contains an end of Λ. Let Λ^+ be the unique unbounded connected component of $\Lambda \cap N(\alpha^+(\delta), k(r))$. We similarly define Λ^-. Let Λ^0 be the closure of $\Lambda \setminus (\Lambda^+ \cup \Lambda^-)$. Thus, Λ^0 is compact.

Now, applying Lemma 5.3, given any $x \in \alpha$, and $t \geq 0$, we can choose δ appropriately so that $\rho(x, \Lambda^-) \geq t$, $\rho(x, \Lambda^+) \geq t$, $\rho(\Lambda^-, \Lambda^+) \geq t$ and $\Lambda^0 \subseteq N(x, u)$, where u depends on r, t, and the parameters of uniformity.

Thus, $\Lambda^{\pm}$ and Λ^0 play similar roles to $A^{\pm}$ and K in the previous discussion. Moreover, we have that $\mathcal{E} = \{\Lambda^-, C_L, \Lambda^+, C_R\}$ is a 4-cycle in X, and that $X = \Lambda^0 \cup \bigcup \mathcal{E}$. However, Λ^0 need not be connected.

6. Consequences for the action of Γ on winding numbers.

Let's begin with a general observation:

Lemma 6.1 : *Suppose Γ is infinite and acts properly discontinuously cocompactly on a metric 2-complex, (X, ρ). The (X, ρ) contains a bi-infinite geodesic.*

Proof : This is a standard fact. Choose a sequence of longer and longer finite geodesic segments, translate their centres to a fixed vertex, and take a diagonal subsequence. $\diamond$

In particular, we see that (X, ρ) is taut. Now suppose that (X, ρ) is planar over the field $\mathbf{F}$. Let $U(\mathbf{F})$ be the multiplicative group of units in the prime subring (i.e. the ring of integers in the prime subfield). As described towards the end of last section, without loss of generality, we may suppose:

(W5) The image of the winding number is precisely the prime subring of $\mathbf{F}$.

It follows that for all $x \in X$, there is an arbitrarily large loop β with $\omega(x, \beta) = 1$.

Now, given $x \in X$, choose a directed oriented uniform bi-infinite path, α through x. Let $Y = N(\alpha, r_0)$, and let $C_L(\alpha)$ and $C_R(\alpha)$ be the left and right deep complements. We can suppose that we have chosen r_0 so that if $\beta \in \mathcal{L}(X)$ with $x \wedge \beta$, then $\deg_\alpha(\beta)$ is defined (using a smaller neighbourhood of α). Thus, $\omega(x, \beta) = \lambda \deg_\alpha(\beta)$, where $\lambda \in U(\mathbf{F})$ depends only on α.

Now, suppose $g \in \Gamma$. We choose the orientation on $g\alpha$ by setting $C_L(g\alpha) = gC_L(\alpha)$ and $C_R(g\alpha) = gC_R(\alpha)$. Now, since $\deg_\alpha(\beta)$ is defined combinatorially, we see that it must be invariant under g, i.e. $\deg_{g\alpha}(g\beta) = \deg_\alpha(\beta)$. It follows that there is some element $\lambda_x(g) \in U(\mathbf{F})$ such that for all β with $x \wedge \beta$, $\omega(gx, g\beta) = \lambda_x(g)\omega(x, \beta)$. Clearly, λ_x does not depend on α.

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Suppose $y \in X$. Let γ be a path connecting x to y, and choose β with $\gamma \wedge \beta$ and $\omega(x, \beta) = \omega(y, \beta) = \omega(\gamma, \beta) = 1$. Now, $g\gamma \wedge g\beta$, so $\omega(gx, g\beta) = \omega(gy, g\beta) = \omega(g\gamma, g\beta)$. Thus, $\lambda_x(g) = \lambda_y(g)$. We can therefore set $\lambda(g) = \lambda_x(g)$. This gives a map $\lambda : \Gamma \longrightarrow U(\mathbf{F})$, so that if $g \in \Gamma$, $x \in X$ and $\beta \in \mathcal{L}(X)$ with $x \wedge \beta$, then $\omega(gx, g\beta) = \lambda(g)\omega(x, \beta)$. Now, it's clear that this λ must be a homomorphism. In summary, we have shown:

Proposition 6.2 : *Suppose that Γ acts properly discontinuously cocompactly on a planar 2-complex (X, ρ). Then, assuming that we have chosen the separation constant sufficiently large, there is a homomorphism $\lambda : \Gamma \longrightarrow U(\mathbf{F})$ such that if $g \in \Gamma$, $x \in X$ and $\beta \in \mathcal{L}(X)$ with $x \wedge \beta$, then $\omega(gx, g\beta) = \lambda(g)\omega(x, \beta)$.* $\diamondsuit$

In particular, we see that the winding number automatically satisfies (W4).

In the case where $\mathbf{F}$ has characteristic 0 so that $U(\mathbf{F}) = \{-1, 1\}$, we derive the following conclusion:

Proposition 6.3 : *Suppose that Γ is a group, and $\mathbf{F}$ is a field of characteristic 0. Suppose that Γ is FP_2 over $\mathbf{F}$, and that $E \subseteq H^2(\Gamma, \mathbf{F}\Gamma)$ is a 1-dimensional Γ-invariant subspace. Then, the subgroup of Γ which fixes E pointwise has index at most 2 in Γ.*

Proof : By Proposition 2.2, Γ admits a free cocompact action on a planar 2-complex. By Lemma 6.1, this complex is taut. Let ω be the winding number arising in this way. The constructions of Section 5 tell us that we can assume property (W5) and that one can define combinatorial degrees of loops. Thus, by Proposition 6.2, we get a homomorphism $\lambda : \Gamma \longrightarrow \{-1, 1\}$, giving property (W4).

On the other hand, we get a character $\theta : \Gamma \longrightarrow \mathbf{F}^{\times}$, arising out of the action of Γ on E, giving us property (W4) directly. Thus, $\theta = \lambda$, and so the subgroup fixing E, namely $\ker \theta = \ker \lambda$, has index at most 2. $\diamondsuit$

In this case, we shall refer to the subgroup fixing E pointwise as the *orientation preserving* subgroup of Γ. At the moment, it is still conceivable that this may depend on the choice of subspace E. A different subspace might give rise to a different winding number. Ultimately however we shall see that, in fact, $H^2(\Gamma; \mathbf{F}\Gamma) = E$.

7. Lifting winding numbers.

We saw in Section 5 how to reduce winding numbers to the prime subring, $\mathbf{A}$, of the field, $\mathbf{F}$. If $\mathrm{char}(\mathbf{F}) = 0$, this is $\mathbf{Z}$, and we are happy. If $\mathrm{char}(\mathbf{F}) = p$, this gives us $\mathbf{Z}_p$, and in order to make the arguments of later sections work, we shall need to lift it to $\mathbf{Z}$. This will require some refinement of the geometric constructions of Section 5. We can allow $\mathbf{F}$ to be any field, though the construction is a bit pointless if its characteristic is 0. Apart from the main result, the constructions of this section are not required elsewhere in this paper.

Let (X, ρ) be a planar metric 2-complex with separation constant r_0. We shall construct our integral winding numbers by identifying a kind of "circular structure" at infinity. The idea can be summarised more formally as follows.

Let $\mathcal{D}$ be the set of compact connected subsets of X, viewed as a directed set under inclusion. We fix a constant, k, to be chosen appropriately. Suppose $K \in \mathcal{D}$, and $Q \supseteq K$ is any complex. Let $I_Q(K)$ be the image of $H_1(N(Q,k)^C; \mathbf{Z})$ in $H_1(K^C; \mathbf{Z})$. Let $I(K) = \bigoplus \{ I_Q(K) \mid Q \supseteq K$ is unbounded$\}$. In other words, $I(K)$ may be defined as the subgroup of $H_1(K^C; \mathbf{Z})$ generated by those loops, β, for which there is some ray, γ, connecting K to infinity, with $\rho(\beta, K \cup \gamma) \geq k$. Let $J(K) = H_1(K^C; \mathbf{Z})/I(K)$.

Note that if $K \subseteq L \in \mathcal{D}$, then the natural map of $H_1(L^C; \mathbf{Z})$ into $H_1(K^C; \mathbf{Z})$ sends $I(L)$ into $I(K)$, and so we get an induced map $J(L) \longrightarrow J(K)$. We thus get an inverse limit system of abelian groups, $(J(K))_{K \in \mathcal{D}}$, and we set J to be the inverse limit.

Now, using Lemma 2.6, it's easy to see that if $x \in K \in \mathcal{D}$, then the map $\omega_x = \omega(x, -)$ vanishes on $I(K)$, and so induces a homomorphism of $J(K)$ to the additive group, $\mathbf{A}$. This map respects the maps $J(L) \longrightarrow J(K)$, so we get a homomorphism $\omega_x : J \longrightarrow \mathbf{A}$. Moreover, there is an element ζ of J defined (up to sign if char$(\mathbf{F}) = 2$) by taking a big sequence, $(\beta_n)_{n \in \mathbf{N}}$ in X with $\omega(x, \beta_n) = 1$ for all n. Now the arguments of Section 5 show fairly easily that this is well-defined: if $K \in \mathcal{D}$, then $\beta_m - \beta_n \in I(K)$ for all sufficiently large m, n. Moreover we see that this element generates J, and that ω_x maps ζ to 1 in $\mathbf{A}$. In summary, this shows that ω_x maps J surjectively to $\mathbf{A}$. Of course, this gives us nothing essentially new. The aim of this section will be to use the combinatorial notion of degree defined in Section 5, to lift this to a surjective map of J to $\mathbf{Z}$, thus showing that J is infinite cyclic. Furthermore, observing that the constructions are uniform (in that the various constants involved depend only on the parameters of X) this gives rise to a rational integer valued winding number satisfying properties (W1), (W2) and (W3).

To this end, it will help if we assume that X admits a cocompact group action. From this it follows immediately that every point of X is a bounded distance from a bi-infinite geodesic. Moreover, given any $r \geq r_0$, we can find some $l_0(r) \geq r$, so that if $x \in X$, there is some $\beta \in \mathcal{L}(N(x, l_0(r)))$ with $\rho(x, \beta) \geq r$ and $\omega(x, \beta) = 1$. (Our arguments work more generally, but it's not worth introducing unnecessary complications here.)

We begin by elaborating on the combinatorial notion of degree defined in Section 5. We shall simplify the discussion by confining our attention to geodesics. Let α be a bi-infinite geodesic, and fix some point, $x_0 \in \alpha$. Let $Y(r) = N(\alpha, r)$ and let $C_L(r)$ and $C_R(r)$ be the left and right deep complements. Given $r \geq 0$, let $\delta(r)$ be the subarc $\alpha \cap N(x_0, 2r)$. Let $\alpha^{\pm}(r) = \alpha^{\pm}(\delta(r))$ and set $A^{\pm}(r) = N(\alpha^{\pm}(r), r)$. There is some $l_1(r)$ such that every shallow complement of $Y(r)$ lies in $N(\alpha, l_1(r))$. We can assume that l_1 is an increasing function of r. Given $\beta \in \mathcal{L}(X)$ with $\rho(x, \beta) \geq l_1(r)$, we set $\deg_{\alpha, r} = d_R^+(\beta) - d_R^-(\beta) = d_L^-(\beta) - d_L^+(\beta)$, where $d_L^{\pm}$ is the number of subpaths in $\beta \setminus (A^+(r) \cup A^-(r))$ which connect $A^{\mp}(r)$ to $A^{\pm}(r)$ and which lie in C_L, and where $d_L^{\pm}$ is defined similarly. Note that this definition also makes sense for any path with endpoints in $A^+(r) \cup A^-(r)$.

Now suppose $s \geq r$. We see that $C_L(s) \subseteq C_L(r)$ and $C_R(s) \subseteq C_R(r)$. Moreover, $\rho(C_L(s), Y(r) \cup C_R(r)) \geq s - r$, and similarly swapping r and s. Suppose γ is a path in $C_R(r)$ with $\partial \gamma = L(r) \cap \gamma$, $\rho(x, \gamma) \geq l_1(s)$ and with initial endpoint in $A^-(r)$ and final endpoint in $A^+(r)$. Now, γ can only cross between $A^-(s)$ and $A^+(s)$ in $C_R(s)$. Clearly there is precisely one more forward crossing than backward crossing, i.e. $\deg_{\alpha, s}(\gamma) = 1$. It now follows that if $\beta \in \mathcal{L}(N(x_0, l_1(s))^C)$, then $\deg_{\alpha, r}(\beta) = \deg_{\alpha, s}(\beta)$. We thus get a fairly robust notion of degree which we shall denote by $\deg_{\alpha}(\beta)$. Of course, we can't use

it directly as a definition of winding number, since, a-priori, it may depend on α. Note that we can assume that we have chosen r_0 large enough so that $\omega(x_0, \beta)$ equals $\deg_\alpha(\beta)$ modulo char(**F**).

The key step in using this notion of degree will be a coarse analogue of the following elementary fact of planar topology.

Lemma 7.1 : *Let β be a (piecewise differentiable) loop in the plane, $\mathbf{R}^2$, with non-zero winding number about the origin $\underline{0} = (0,0) \notin \text{image}(\beta)$. Then, there is a loop, γ, with winding number 1 about $\underline{0}$, with $\text{image}\,\gamma \subseteq \text{image}\,\beta$.*

Proof : Consider the boundary of the component of $\mathbf{R}^2 \setminus \text{image}(\beta)$ containing $\underline{0}$. $\qquad\qquad \diamond$

We aim to prove:

Lemma 7.2 : *There is a constant, $l_5 \geq r_0$ (depending only on the parameters of X) such that if α is a bi-infinite geodesic in X, $x_0 \in \alpha$, and $\beta \in \mathcal{L}(N(x_0, 2l_5)^C)$ with $\deg_\alpha(\beta) \neq 0$, then there is a loop $\gamma \subseteq N(\beta, l_5)$ with $\deg_\alpha(\gamma) = 1$.*

Let's fix a bi-infinite geodesic α, and a point $x_0 \in \alpha$. Let $A^\pm = A^\pm(2r_0)$, $Y = Y(2r_0)$, $C_L = C_L(2r_0)$ and $C_R(2r_0)$ etc. By an *L-path*, we mean a path $\beta \subseteq Y \cup C_L$ with endpoints in α, so that $\beta \cap Y$ consists of two geodesic segments each of length $2r_0$, and with $\beta \cap C_L$ a non-empty subpath. Thus, $\partial\beta = \beta \cap \alpha$. We similarly define an *R-path* where $\beta \subseteq Y \cup C_R$. We say that a path is an *LR-path* if it is either an L-path or an R-path.

Given a sequence, $\beta_1, \ldots, \beta_n$ of LR-paths, we define $\sigma(\beta_1, \ldots, \beta_n)$ to be the loop $\beta_1 \cup \delta_1 \cup \beta_2 \cup \delta_2 \cup \cdots \cup \beta_n \cup \delta_n$, where δ_i is the subpath of α connecting the final point of β_i to the initial point of β_{i+1}. We refer to a loop arising in this way as a *σ-loop*. Now, each shallow complement of Y lies inside $N(\alpha, l_2)$ where $l_2 = l_1(2r_0)$. From this, it follows easily that if $\beta \in \mathcal{L}(X)$ with $\rho(x_0, \beta) \geq l_2$, then there is some σ-loop, σ, lying in $N(\beta, l_2)$ with $\deg_\alpha(\beta) = \deg_\alpha(\sigma)$. To see how to obtain σ, consider a component, δ, of β with the union of Y and all the shallow complements. Thus, δ is a subarc, with endpoints, $x, y \in \partial Y$. We replace this with an arc, δ', which goes directly from x to α, runs along a segment of α, and then returns to y. The arcs connecting x and y to α we can take to be geodesics, each of length $2r_0$. Thus, $\delta' \subseteq N(\delta, l_0)$. We replace each such subarc, δ, by an arc, δ', in this way. Note that $\deg_\alpha(\sigma)$ is the number of positive R-paths minus the number of negative R-paths making up σ. By a "positive" R-path we mean one connecting α^- to α^+, whereas a "negative" R-path connects α^+ to α^-. (Of course an R-path might be neither.)

Given $x, y \in \alpha$, we write $x < y$ if $x = \alpha(t)$ and $y = \alpha(u)$ with $t < u$. If $x < y < z$, we say that y *interlocks* $\{x, z\}$. If $x < y < z < w$, we say that $\{x, z\}$ *interlocks* $\{y, w\}$. If γ is an LR-path and $x \in \alpha$, we say that x *interlocks* γ if it interlocks $\partial\gamma$. If γ and δ are LR-paths, we say that γ *interlocks* δ if $\partial\gamma$ interlocks $\partial\delta$, and γ and δ are either both L-paths or both R-paths.

Now let $l_3 = l_0(l_2 + 2r_0)$. Thus, if $x \in \alpha$, then (by the definition of the function l_0, some loop, β with $\omega(x_0, \beta) \neq 0$, with $\beta \subseteq N(x, l_3) \cap N(x, l_2 + 2r_0)^C$. From this it follows that there is some L-path, δ, interlocking x, with $\delta \subseteq N(x, l_2)^C \cap N(x, l_3)$, and a similar R-path, ϵ. Note in particular that $\rho(x_0, \partial\delta) \geq l_2$ and $\rho(x_0, \partial\epsilon) \geq l_2$.

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We shall need the following observation:

Lemma 7.3 : *Suppose β is an L-path and that γ is an R-path with $\rho(\partial\beta, \partial\gamma) \geq 5r_0$. Then $\rho(\beta, \gamma) \geq r_0$.*

Proof : Let $C_L^0 = C_L(r_0)$, $C_R^0 = C_R(r_0)$ and $Y^0 = Y(r_0)$. Then, $\rho(C_L, Y^0 \cup C_R^0) \geq r_0$ and $\rho(C_L, Y^0 \cup C_R^0) \geq r_0$. Now, $\rho(\beta \cap Y, \gamma \cap Y) \geq 5r_0 - 2(2r_0) = r_0$. Since $\beta \subseteq Y^0 \cup C_R^0$ and $\gamma \subseteq Y^0 \cup C_R^0$, we have $\rho(\beta \cap C_L, \gamma) \geq r_0$ and $\rho(\gamma \cap C_R, \beta) \geq r_0$. This covers all possibilities, and so $\rho(\beta, \gamma) \geq r_0$ as claimed. $\diamondsuit$

Now let $l_4 = l_3 + 5r_0$.

Lemma 7.4 : *Suppose β and γ are interlocking LR-paths. Then $\rho(\beta, \gamma) \leq l_4$.*

Proof : Suppose that β and γ are both L-paths and that $\rho(\beta, \gamma) > l_4$. Let $\partial\beta = \{x, z\}$ and $\partial\gamma = \{y, w\}$. We can assume that $x < y < z < w$. Note, in particular, that $\rho(y, w) > \rho(z, w) > l_4$. By the observation before Lemma 7.3, we can find an R-path, δ, with $\partial\delta = \{a, b\}$ with $a < y < b$, and $l_2 \leq \rho(y, a) \leq l_3$ and $l_2 \leq \rho(y, b) \leq l_3$. Now, $\rho(b, w) \geq \rho(y, w) - \rho(y, b) \geq l_4 - l_3 = 5r_0$. Thus, $\rho(w, \partial\delta) \geq 5r_0$, and so $\rho(\partial\gamma, \partial\delta) \geq 5r_0$. By Lemma 7.3, $\rho(\gamma, \delta) \geq r_0$. Also $\rho(\beta \cup \delta, \alpha^+(w)) \geq r_0$ (where $\alpha^+(w)$ is the positive ray of α based at w).

Now let $\sigma = \sigma(\beta, \delta)$. We see that $\rho(\sigma, \gamma \cup \alpha^+(w)) \geq r_0$. But, $\gamma \cup \alpha^+(w)$ is a path connecting y to infinity. Therefore, by Lemma 3.6, $\omega(y, \sigma) = 0$. However, $\deg_\alpha(\sigma) = 1$, where $\deg_\alpha$ here represents the combinatorial degree about y. By Proposition 5.8, $\omega(y, \sigma) = \deg_\alpha(\sigma)$. We therefore derive the contradiction that $0 = 1$. $\diamondsuit$

Proof of Lemma 7.2 : Let $l_5 = l_2 + l_4$. Suppose that $x_0 \in X$ and $\beta \in \mathcal{L}(N(x_0, 2l_5)^C)$ with $\deg_\alpha(\beta) \neq 0$. As we have observed, there is a σ-loop, $\sigma = \sigma(\beta_1, \ldots, \beta_n) = \beta_1 \cup \delta_1 \cup \cdots \cup \beta_n \cup \delta_n$ in $N(\beta, l_2)$ with $\deg_\alpha(\sigma) = \deg_\alpha(\beta)$. We now use Lemma 7.4 to shortcut σ and obtain the desired loop, γ. The argument is essentially combinatorial, but it is easier to express it geometrically as follows.

Let $\alpha' = \mathbf{R} \times \{0\} \subseteq \mathbf{R}^2$. Let $C_L' = \mathbf{R} \times [0, \infty)$ and $C_R' = \mathbf{R} \times (-\infty, 0]$. We parametrise α by arc-length so that $\alpha(0) = x_0$. Given an L-path, ϵ, let ϵ' be the semicircle in C_L' with endpoints $(t, 0)$ and $(u, 0)$, where $\partial\epsilon = \{\alpha(t), \alpha(u)\}$. We similarly define ϵ' for an R-path ϵ. Now, let σ' be the loop $\beta_1' \cup \delta_1' \cup \cdots \cup \beta_n' \cup \delta_n'$, where $\delta_i' \subseteq \alpha'$ is the segment connecting the final point of β_i' with the initial point of β_{i+1}'. Clearly, the winding number of σ' about the origin equals $\deg_\alpha(\sigma)$, which is non-zero. Thus, by Lemma 7.1, there is a loop, γ', with image$(\gamma') \subseteq$ image(σ'), and with winding number 1 about the origin.

We now use this to construct the loop γ by following the corresponding segments of β_i and δ_i. The only complication is that γ' may cross between different semicircles, β_i' and β_j'. However in this case, the corresponding paths β_i and β_j interlock. Thus, Lemma 7.4 allows us to cross from β_i to β_j along a path of length at most l_4. Thus, $\gamma \subseteq N(\sigma, l_4) \subseteq N(\beta, l_4 + l_2) = N(\beta, l_5)$. Since $\rho(x_0, \gamma) \geq 2l_5 - l_5 = l_5 > l_2$, it follows that $\deg_\alpha(\gamma)$ is defined and equal to 1. $\diamondsuit$

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Now let $k = l_5 + r_0$.

Corollary 7.5 : *Suppose that $\beta \in \mathcal{L}(X)$, with $\rho(x, \beta) \geq 2k$. Suppose that there is a path ϵ connecting x_0 to infinity with $\rho(\epsilon, \beta) \geq k$. Then $\deg_\alpha(\beta) = 0$.*

Proof : Suppose not. Let $\gamma \subseteq N(x, l_5)$ be the loop given by Lemma 7.2. Thus $\omega(x_0, \gamma) = \deg_\alpha(\gamma) = 1$. Also $\rho(\epsilon, \gamma) \geq k - l_5 = r_0$. Thus, by Lemma 3.6, $\omega(x_0, \gamma) = 0$. Therefore $0 = 1$. $\diamond$

Lemma 7.6 : *The map $\deg_\alpha$ extends to a homomorphism, $\deg_\alpha : H_1(N(x_0, l_2)^C; \mathbf{Z}) \longrightarrow \mathbf{Z}$.*

Proof : Clearly, if $\beta \in \mathcal{L}(N(x_0, l_2)^C)$ has diameter at most r_0, then $\deg_\alpha(\beta) = 0$. We need also to verify that if $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{L}(N(x_0, l_2)^C; \mathbf{Z})$ form a theta curve then $\deg_\alpha(\gamma_1) + \deg_\alpha(\gamma_2) + \deg_\alpha(\gamma_3) = 0$. We can write $\gamma_i = \beta_i \cup -\beta_{i+1}$ where $\beta_1, \beta_2, \beta_3$ are paths with the same endpoints. By attaching paths to these endpoint if necessary, we can assume that both endpoints lie in α. This means that $\deg_\alpha(\beta_i)$ is defined, and that $\deg_\alpha(\gamma_i) = \deg_\alpha(\beta_i) - \deg_\alpha(\beta_{i+1})$. $\diamond$

Let $\mathcal{D}(x_0) = \{K \in \mathcal{D} \mid N(x_0, l_2) \subseteq K\}$. Note that $\mathcal{D}(x_0)$ is cofinal in $\mathcal{D}$. Also, if $K \in \mathcal{D}$, then, by Lemma 7.6, we have a homomorphism $\deg_\alpha : H_1(K^C; \mathbf{Z}) \longrightarrow \mathbf{Z}$. By Lemma 7.5, this is identically zero on a generating set of $I(K)$. We thus have:

Lemma 7.7 : *If $K \in \mathcal{D}(x_0)$, then the map $\deg_\alpha : H_1(K^C; \mathbf{Z}) \longrightarrow \mathbf{Z}$ is identically zero on $I(K)$.* $\diamond$

We thus get a homomorphism, also denoted $\deg_\alpha$, from $J(K)$ to $\mathbf{Z}$ for all such K. Moreover, $\deg_\alpha$ clearly commutes with all maps in the direct limit system, so we get an induced homomorphism $\deg_\alpha : J \longrightarrow \mathbf{Z}$.

Now, let $(\beta_n)_n$ be a big sequence in X, with $\deg_\alpha(\beta_n) = 1$ for all n. Suppose that $K \in \mathcal{D}(x_0)$. There is some $r \geq 0$ such that $K \subseteq N(x_0, r) \subseteq N(\alpha, r)$. Now, the argument of Lemma 4.3 shows that for all sufficiently large m and n, $\beta_m - \beta_n \in I(K)$. Thus, the sequence defines an element, $\zeta_K \in J(K)$. Since $\mathcal{D}(x_0)$ is cofinal, we thus get an element, $\zeta \in J$. By definition, $\deg_\alpha(\zeta) = 1$. In particular, the map $\deg_\alpha : J \longrightarrow \mathbf{Z}$ is surjective.

Again, the argument of Lemma 4.3 shows that there is some $L \in \mathcal{D}(x_0)$, containing K, such that if $\beta \in \mathcal{L}(L^C)$, then $\beta - (\deg_\alpha(\beta))\beta_n \in I(K)$, for all sufficiently large n. Thus β is represented by $(\deg_\alpha(\beta))\zeta_K$ in $J(K)$. In other words, the image of $J(L)$ in $J(K)$ is generated by ζ_K. This shows that J is generated by ζ.

In particular, we have shown:

Proposition 7.8 : *The group J is infinite cyclic.* $\diamond$

We are assuming that X admits a cocompact group action, so that every point lies a bounded distance from a bi-infinite geodesics, and all the above constructions are uniform. We see that there are constants, $l_7 > l_6 > l_2$, such that if $x \in X$ and $L \in \mathcal{D}$ with

$L \supseteq N(x_0, l_7)$, then the image of $J(L)$ in both $J(N(x, l_6))$ and $J(N(x, l_6 + 1))$ is infinite cyclic, with the generator naturally identified with ζ.

Now suppose that $\beta \in \mathcal{L}(N(x, l_7)^C)$. Then, β is represented by some multiple, $\tilde{\omega}(x, \beta)$, of ζ. We check that this satisfies properties (W1)–(W3). Now, (W1) and (W3) are immediate. To see (W2), suppose that $x, y \in X$ are adjacent, and suppose $\rho(\{x, y\}, \beta) \geq l_7$. Let $L = N(\{x, y\}, l_7)$. Now, the image of $J(L)$ in $J(N(\{x, y\}, l_6))$ is infinite cyclic, so β is represented by some multiple of ζ in $J(N(\{x, y\}, l_6))$. This multiple must be equal to both $\tilde{\omega}(x, \beta)$ and $\tilde{\omega}(y, \beta)$. $\diamond$

Note that $\omega(x, \beta)$ equals $\tilde{\omega}(x, \beta)$ modulo char$(\mathbf{F})$. If char$(\mathbf{F}) = 0$, we have achieved nothing. If char$(\mathbf{F}) = p$, then, at the cost of increasing the separation constant (from r_0 to l_7) we have lifted the winding number from $\mathbf{Z}_p$ to $\mathbf{Z}$.

8. Orders and cyclic orders.

Let $(T, \leq)$ be a totally ordered set. Suppose that $g : T \longrightarrow T$ is an order automorphism. We say that g is *positive* if $gx > x$ for all $x \in T$, and *negative* if $gx < x$ for all $x \in T$. We say that g is *archimedean* if, for all $x, y, z \in T$, $\{n \in \mathbf{Z} \mid x < g^n z < y\}$ is finite. Note that an archimedean map is either positive or negative. Moreover, g is archimedean if and only if g^n is archimedean for all non-zero $n \in \mathbf{Z}$.

Suppose a group, Γ, acts by automorphism on T. We say that the action is *archimedean* if every non-identity element of Γ is archimedean. Clearly, Γ acts freely on T. In fact, Γ must be abelian, by the result of Hölder, Frege and Huntington (see for example, [ADN] for a discussion).

Now, if T is countable, we can embed T canonically in the real line, or more precisely in a totally ordered set which is order isomorphic to the real line. (For example, first embed T as $T \times \{0\}$ in $T \times \mathbf{Q}$ with the antilexicographic order. This a countable dense order, so we can complete it to give the reals.) Any order automorphism of T extends canonically to an orientation preserving homeomorphism of $\mathbf{R}$. Thus a group action on T extends to a group action on $\mathbf{R}$. If the action on T is archimedean, then so is the action on $\mathbf{R}$. Conversely, any orientation preserving group acting freely on $\mathbf{R}$ is archimedean, hence abelian (in fact a subgroup of the additive reals).

Now let Θ be a set. By a *cyclic order* on Θ we mean a function from the set of distinct ordered triples, $\{(x, y, z) \in \Theta^3 \mid x \neq y \neq z \neq x\}$, of Θ, to $\{-1, 1\}$, satisfying $\sigma(x, y, z) = \sigma(y, z, x) = -\sigma(y, x, z)$ for all distinct $x, y, z \in \Theta$, and if $\sigma(x, y, z) = \sigma(x, z, w) = 1$, then $\sigma(x, y, w) = 1$. Clearly the circle, S^1, admits such an order. One can show inductively that any finite subset of Θ can be embedded in the circle, so that the standard cyclic order restricts to the given one. In fact, in a manner analogous to that described for total orders (i.e. first embedding every point in a copy of the rationals), we can canonically embed a countable cyclically ordered set in (a set cyclically order isomorphic to) S^1. Thus, every automorphism of Θ extends canonically to a homeomorphism of S^1.

Suppose the infinite cyclic group, $\mathbf{Z}$, has an archimedean action on the totally ordered set, T. Given $x \in T$ and $n \in \mathbf{Z}$, we shall write $x + n$ for $g^n x$, where g is the positive generator of the action. Let Θ be the quotient, and $\pi : T \longrightarrow \Theta$ be the quotient map.

Now, Θ admits a cyclic order. One way to define this is to say that $\sigma(x, y, z) = 1$ if $\tilde{x} < \tilde{y} < \tilde{z} < \tilde{x} + 1$, for suitable lifts of $\tilde{x}, \tilde{y}, \tilde{z}$ of x, y, z to T. Alternatively, embed T canonically in $\mathbf{R}$, extend the action of $\mathbf{Z}$ to $\mathbf{R}$, and take the induced cyclic order on $\Theta = T/\mathbf{Z}$ from $S^1 = \mathbf{R}/\mathbf{Z}$.

Conversely, if (Θ, σ) is a cyclically ordered set, we can express Θ as a quotient $T/\mathbf{Z}$, where T is a totally ordered set with an archimedean action of $\mathbf{Z}$. (For example, embed Θ in S^1, and lift to $\mathbf{R}$.) Any automorphism, g, of Θ lifts to an automorphism, $\tilde{g}$, of T. Now, $\tilde{g}$ commutes with the action of $\mathbf{Z}$. Moreover, if $\tilde{g}'$ is another lift, then $\tilde{g}' \circ \tilde{g}^{-1}$ has the form $[x \mapsto x + n]$ for some $n \in \mathbf{Z}$.

In this way, we can define the *rotation number*, $\operatorname{rot}(g) \in \mathbf{R}/\mathbf{Z}$, of an automorphism, g, of Θ in the usual dynamical fashion (as for maps of the circle). In the case of interest here, namely where g has finite order, then $\operatorname{rot}(g) \in \mathbf{Q}/\mathbf{Z}$. This can be described explicitly as follows. Let $\tilde{g}$ be a lift of g to T. If $g^n = 1$, then $\tilde{g}^n = [x \mapsto x + m]$ for some $m \in \mathbf{Z}$, and we set $\operatorname{rot}(g) = m/n$. Note that for any $p \in \mathbf{Z}$, $\operatorname{rot}(g^p) = p \operatorname{rot}(g)$. If Γ acts on Θ, we get a map $\operatorname{rot} : \Gamma \longrightarrow \mathbf{R}/\mathbf{Z}$. In general this need not be a homomorphism. However, restricted to any cyclic subgroup, it is. Moreover, it is conjugacy invariant: $\operatorname{rot}(hgh^{-1}) = \operatorname{rot}(g)$ for any $h \in \Gamma$.

Note that if Γ is a torsion group acting effectively on Θ, (i.e. every element has finite order and only the identity acts trivially), then the induced action on the circle also has this property, and so the action must be free. Note also that any finite group acting freely on the circle is cyclic.

We note the following well known result:

Proposition 8.1 : *Any group acting freely on the circle is abelian.*

Proof : One way to see this is to represent the group as a quotient, $G/\mathbf{Z}$, of a group G which acts freely on the real line by an infinite cyclic group. From an earlier observation, G is abelian. $\diamondsuit$

Putting this together with the previous observation, we conclude:

Corollary 8.2 : *Suppose that Γ is a torsion group acting effectively on a countable cyclically ordered set. Then Γ is locally cyclic.* $\diamondsuit$

Thus, Γ is abelian, and in this case, $\operatorname{rot} : \Gamma \longrightarrow \mathbf{Q}/\mathbf{Z}$ is a homomorphism.

9. Rotational parts.

Suppose X is a planar 2-complex, and that ω is an integer valued winding number satisfying properties (W1)–(W3), with separation constant, r_0.

Let $\mathcal{M}$ be the set of pairs (M, K), where $K \subseteq X$ is compact and connected, $M \subseteq X$ is connected with M^C compact, and with $M \wedge K$ (i.e. $\rho(M, K) \geq r_0$). Given $(M, K), (M', K') \in \mathcal{M}$, we write $(M, K) \geq (M', K')$ to mean that $M \subseteq M'$ and $K' \subseteq K$. Given that X is one-ended (by Proposition 3.8) it is clear that $\mathcal{M}$ is a directed set with this order.

We shall frequently suppress explicit mention of the set K. Thus, we speak of an element $M \in \mathcal{M}$, and assume that it has associated with it a compact set K, denoted K_M, as described above. We have a cocycle, $\mu_M : H_1(M; \mathbf{Z}) \longrightarrow \mathbf{Z}$ defined on the generating set $\mathcal{L}$ by $[\beta \mapsto \omega(K_M, \beta)]$. (The primary function of K_M is to define this cocycle.) If $M, M' \in \mathcal{M}$, and $M' \geq M$, then μ_M restricted to $H_1(M'; \mathbf{Z})$ agrees with $\mu_{M'}$.

Given $M \in \mathcal{M}$, we write $\tilde{M}$ for the infinite cyclic cover of M given by μ_M. The *positive* generator of this action corresponds to a loop $\beta \in \mathcal{L}(M)$ with $\mu_M(\beta) = 1$. We shall denote the action of this generator by $[x \mapsto x + 1]$, and of its mth power by $[x \mapsto x + m]$. Thus, if $x \in M$ and γ is any path connecting x to $x + m$ in M, then γ projects to a loop, δ, in M with $\mu_M(\delta) = 1$. If $M' \geq M$, then we can identify $\tilde{M}'$ as a subset of $\tilde{M}$.

Suppose now that g is a finite order orientation preserving automorphism of X. Recall that this means that $\omega(gx, g\beta) = \omega(x, \beta)$ for all $x \in X$ and $\beta \in \mathcal{L}(X)$ with $x \wedge \beta$. Let $\mathcal{M}(g)$ be the set of g-invariant elements of $\mathcal{M}$, i.e. $\{M \in \mathcal{M} \mid gM = M, gK_M = K_M\}$. Clearly in this case, $\mu_M(g\beta) = \mu_M(\beta)$ for all β.

Lemma 9.1 : $\mathcal{M}(g)$ *is cofinal in* $\mathcal{M}$.

Proof : Given $M \in \mathcal{M}$, let K be any compact g-invariant set containing $\bigcup_n g^n K_M$. Since X is one-ended, there is some connected $P \subseteq X$ with P^C compact, and with $K \wedge P$. Let $Q = \bigcap_n g^n P$. Set $K_Q = K$. Thus $Q \in \mathcal{M}(g)$, and $Q \geq M$. $\diamond$

Suppose g is orientation preserving of order n. Choose any $M \in \mathcal{M}(g)$. Thus, g lifts to a map $\tilde{g} : \tilde{M} \longrightarrow \tilde{M}$, so that $\tilde{g}^n$ projects to the identity. Note that $\tilde{g}$ commutes with the generator $h = [x \mapsto x + 1]$. (Since the commutator, $[\tilde{g}, h]$, projects to the identity, it must equal h^p for some p, and since $[\tilde{g}^n, h] = 1$, we deduce that $p = 0$.) Now, $\tilde{g}$ has the form $[x \mapsto x + m]$ for some $m \in \mathbf{Z}$. Set $\mathrm{rot}_M(g) = m/n \in \mathbf{Q}/\mathbf{Z}$. This is clearly independent of the choice of lift, $\tilde{g}$. Now, if $P \in \mathcal{M}(g)$ with $P \geq M$, then $\tilde{g}|\tilde{P}$ is a lift of $g|P$, so we see that $\mathrm{rot}_P(g) = \mathrm{rot}_M(g)$. We thus get a well defined number $\mathrm{rot}(g) \in \mathbf{Q}/\mathbf{Z}$. Note that $\mathrm{rot}(g) = 0$ if and only if some lift of g has finite order (equal to n).

Definition : We call $\mathrm{rot}(g)$ the *rotational part* of the finite order orientation preserving automorphism g.

A more direct way to define the rotational part is as follows. Choose any compact connected subset $K \subseteq X$ with $gK = K$, and let β be any path with $K \wedge \beta$ which connects some point $x \in X$ to gx. Then $\mathrm{rot}(g) = \omega(K, \gamma)$ where γ is the loop $\beta \cup g\beta \cup g^2\beta \cup \cdots \cup g^{n-1}\beta$.

Note that if g, h are finite order orientation preserving automorphisms and $p \in \mathbf{Z}$, then $\mathrm{rot}(g^p) = p\,\mathrm{rot}(g)$ and $\mathrm{rot}(gh) = \mathrm{rot}(hg)$.

Given an automorphism, g, define a map $D_g : X \longrightarrow [0, \infty)$ by $D_g(x) = \rho(x, gx)$.

Lemma 9.2 : *Suppose g is orientation preserving with finite order and with $\mathrm{rot}(g) \neq 0$. Then $D_g(x) \to \infty$ as $x \to \infty$ (i.e. $\{x \in X \mid D_g(x) \leq r\}$ is compact for all $r \geq 0$).*

Proof : Let the order of g be n, and suppose $r \geq 0$. Now, using Lemma 3.4, we see that there is some $M \in \mathcal{M}$, such that if $\gamma \in \mathcal{L}(M)$ with length$(\gamma) \leq nr$, then $\mu(\gamma) = 0$. Suppose $x \in M$ with $D_g(x) \leq r$. Let β be an arc of length at most r connecting x to gx. Let $\gamma = \beta \cup g\beta \cup \cdots \cup g^{n-1}\beta$. Then length$(\gamma) \leq nr$, and so rot$(g) = \mu(\gamma) = 0$. $\diamondsuit$

10. Constructing orders.

In this section, we describe how certain classes of subsets of a planar 2-complex, (X, ρ), have a natural cyclic order.

Recall the definition of the directed set, $\mathcal{M}$, from Section 9. Suppose $M \in \mathcal{M}$. A *generating loop* is a loop $\gamma \in \mathcal{L}(M)$ with $\mu_M(\gamma) = 1$. This lifts to a bi-infinite path, $\tilde{\gamma} \in \tilde{M}$. By a *long path* in $\tilde{M}$, we mean a path which remains a bounded distance from the lift of some (hence every) generating loop in M. Note that if we alter a long path over any finite subpath, then it remains long.

We say that a connected subset, $A \subseteq M$, is *separating* if it intersects every generating loop but does not contain any generating loop. (As usual, we are tacitly assuming that A is a subcomplex of M.) The preimage of A in $\tilde{M}$ is a disjoint union of sets of the form $\tilde{A} + n$ for $n \in \mathbf{Z}$, where $\tilde{A}$ is a connected *lift* of A to $\tilde{M}$. One can show that $\tilde{A}$ meets every long path in $\tilde{M}$, and that if $m, n \in \mathbf{Z}$, then any path connecting $\tilde{A} - m$ to $\tilde{A} + m$ meets $\tilde{A}$. We shall omit proof of these statements, since they follow easily in the specific cases where separating sets arise.

For example, suppose $\mathcal{A}$ is a p-cycle with $\bigcup \mathcal{A} = M$ (as defined in Section 4), so that μ_M is identically zero on $A \cup B$ for any $A, B \in \mathcal{A}$. We can lift $\mathcal{A}$ to an "∞-cycle", $\tilde{\mathcal{A}}$, in $\tilde{M}$ (i.e. $\Omega(\tilde{\mathcal{A}})$ is a graph homeomorphic to the real line). It follows easily that each $A \in \mathcal{A}$ is separating, and satisfies the properties of the previous paragraph.

A particular example of this is where α is a bi-infinite geodesic (or uniform path) and $x \in \alpha$. In this case, for any $r \geq r_0$, we get a 4-cycle, $\mathcal{E} = \{\Lambda^-, C_L, \Lambda^+, C_R\}$ as defined at the end of Section 5. (This depends on a "radius" parameter, r, though this will not matter in the construction we use. For definiteness, we can set $r = r_0$.) Let $M = \bigcup \mathcal{E}$. Then $X = \Lambda^0 \cup M$, where Λ^0 is compact, and indeed has diameter bounded (in terms of r and the functions of uniformity). Let $K_M = \{x\}$. This gives us an element of $\mathcal{M}$, which we denote by $M(\alpha, x)$. Since $\mathcal{E}$ is a 4-cycle, we see that the sets Λ^+ and Λ^- are both separating in M. Note that they are also both one-ended. We shall denote them by $\Lambda^{\pm}(\alpha, x)$.

Suppose now that $\mathcal{A}$ is a collection of one-ended subsets of X. Suppose that to each $A \in \mathcal{A}$ there is associated some $M_A \in \mathcal{M}$ such that $A \subseteq M_A$, and A is separating in M_A. We shall write μ_A for the cocycle μ_{M_A}. Let's assume in addition that:

(∗) Given distinct $A, B \in \mathcal{A}$, there is some $M \in \mathcal{M}$ such that $M \cap A \cap B = \emptyset$.

We shall put a cyclic order on the set $\mathcal{A}$.

Given $A \in \mathcal{A}$, let $\mathcal{S}(A)$ be the set of lifts of A to M_A. If $P \in \mathcal{M}$ with $M_A \leq P$, then we identify $\tilde{P}$ as a subset of $\tilde{M}_A$, and write $\mathcal{S}(A, P) = \{S \cap \tilde{P} \mid S \in \mathcal{S}(A)\}$. (Note that the elements of $\mathcal{S}(A, P)$ need not be connected.)

Now, let $\mathcal{F}(\mathcal{A})$ be the set of finite subsets of $\mathcal{A}$. This can be viewed as a directed set under inclusion. Suppose $\mathcal{B} \in \mathcal{F}(\mathcal{A})$. Since $\mathcal{M}$ is a directed set, we can find some $P \in \mathcal{M}$ with $M_A \leq P$ for all $A \in \mathcal{B}$. We can view $\tilde{P}$ as being (simultaneously) a subset of $\tilde{M}_A$ for each $A \in \mathcal{B}$. Set $\mathcal{S}_P(\mathcal{B}) = \bigcup_{A \in \mathcal{B}} \mathcal{S}(A, P)$. Thus, $\mathcal{S}_P(\mathcal{B})$ is a **Z**-invariant collection of subsets of $\tilde{P}$.

Lemma 10.1 : *Any long path in $\tilde{P}$ meets every element of $\mathcal{S}_P(\mathcal{B})$.*

Proof : If it misses a lift of $A \in \mathcal{B}$, then it misses it also in $\tilde{M}_A$ contrary to the assumption that A is separating in M_A. $\diamond$

Note that using property $(*)$, we can choose P so that $P \cap A \cap B = \emptyset$ for all distinct $A, B \in \mathcal{B}$. In this case, the elements of $\mathcal{S}_P(\mathcal{B})$ will all be disjoint.

Since each element of $\mathcal{B}$ is one-ended, we can find $Q \in \mathcal{M}$ with $Q \geq P$, so that if $A \in \mathcal{B}$, then $A \cap Q$ lies in the unbounded component of $A \cap P$. Lifting to $\tilde{Q} \subseteq \tilde{P}$, we see that if $S \in \mathcal{S}_Q(\mathcal{B})$, then $S = S_P \cap \tilde{Q}$ for some $S_P \in \mathcal{S}_P(\mathcal{B})$, and that any two points of S are connected by a path in S_P.

We aim to put a **Z**-invariant archimedean total order on $\mathcal{S}_Q = \mathcal{S}_Q(\mathcal{B})$. This then descends to a cyclic order on $\mathcal{S}_Q(\mathcal{B})/\mathbf{Z}$ which can be canonically identified with $\mathcal{B}$.

To this end, let β be a long path in Q, and write $<$ for the order of points on β. (If β is not embedded, we should more properly pull back to the domain of β, but this will only confuse the notation.) If $S \in \mathcal{S}_Q$, then $\beta \cap S$ is a nonempty compact set, and we write $\text{init}_\beta(S)$ and $\text{final}_\beta(S)$ respectively for the initial and final points of $\beta \cap S$. Given $R, S \in \mathcal{S}_Q$, we write $R <_\beta S$ to mean that $\text{init}_\beta(R) < \text{init}_\beta(S)$. Clearly, $(\mathcal{S}_Q, <_\beta)$ is a discrete total order (i.e. all intervals are finite).

Lemma 10.2 : *If $R, S \in \mathcal{S}_Q$, then $R <_\beta S$ if and only if $\text{final}_\beta(R) < \text{final}_\beta(S)$.*

Proof : By symmetry, it's enough to prove "only if". Suppose, to the contrary, that $\text{final}_\beta(S) < \text{final}_\beta(R)$. Now, $S = S_P \cap \tilde{Q}$ and $R = R_P \cap \tilde{Q}$. Connect $\text{init}_\beta(R)$ to $\text{final}_\beta(R)$ by a path $\epsilon \subseteq R_P$, and let $\gamma = \beta^-(\text{init}_\beta(R)) \cup \epsilon \cup \beta^+(\text{final}_\beta(R))$. Thus, γ is a long path in $\tilde{P}$, and $\gamma \subseteq Q \cup R_P$. Since $S_P \cap R_P = \emptyset$, we see that $\gamma \cap S_P = \emptyset$ contrary to Lemma 10.1. $\diamond$

Lemma 10.3 : *Suppose β and γ are long paths in $\tilde{Q}$ and $R, S \in \mathcal{S}_Q$. Then $R <_\beta S$ if and only if $R <_\gamma S$.*

Proof : Suppose, for contradiction, that $R <_\beta S$ and $S <_\gamma R$. Let ϵ be a path in R_P from $\text{init}_\beta R$ to $\text{final}_\beta(R)$. Let $\delta = \beta^-(\text{init}_\beta(R)) \cup \epsilon \cup \gamma^+(\text{final}_\gamma(R))$. Thus, δ is a long path and $\delta \subseteq \tilde{Q} \cup R_P$. By Lemma 10.2, $\gamma^+(\text{final}_\gamma(R)) \cap S = \emptyset$. It follows that $\delta \cap S_P = \emptyset$ contradicting Lemma 10.1. $\diamond$

This shows that the order $<_\beta$ is independent of the choice of long path β in $\tilde{Q}$. We thus get an order, denoted $<_Q$, on $\mathcal{S}_Q(\mathcal{B})$. Taking β to be the lift of a generating loop, we see immediately that the order is **Z**-invariant, and that $S <_Q S + n$ for all $n > 0$.

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This is archimedean, and so gives rise to a cyclic order, $\sigma_{\mathcal{B},Q}$ on $\mathcal{S}_Q(\mathcal{B})/\mathbf{Z}$ and hence on $\mathcal{B}$. Clearly, if $Q' \in \mathcal{M}$, with $Q' \geq Q$, then $\sigma_{\mathcal{B},Q} = \sigma_{\mathcal{B},Q'}$. Since the set of such Q is cofinal in $\mathcal{M}$, we get a well-defined cyclic order, $\sigma_{\mathcal{B}}$, on $\mathcal{B}$. There was some choice involved in the identification of $\tilde{P}$ as a subset of the various $\tilde{M}_A$ for $A \in \mathcal{B}$. However the identifications are canonical up to the action of $\mathbf{Z}$, and so any set of choices will give rise to the same cyclic order on $\mathcal{B}$.

Now, if $\mathcal{B}, \mathcal{C} \in \mathcal{F}(\mathcal{A})$ with $\mathcal{C} \subseteq \mathcal{B}$, then working with $\mathcal{B}$ and restricting to $\mathcal{C}$, we see that $\sigma_{\mathcal{B}}$ restricted to $\mathcal{C}$ agrees with $\sigma_{\mathcal{C}}$. We thus get a direct limit system of cyclic orders giving rise to a cyclic order, σ, on $\mathcal{A}$. Formally, we can define σ by $\sigma(A, B, C) = \sigma_{\{A,B,C\}}(A, B, C)$.

In retrospect, we see that we can define the cyclic order more directly as follows. Given $A, B, C \in \mathcal{A}$, choose $P \in \mathcal{M}$ with $P \geq M_A, M_B, M_C$ and so that $A \cap P, B \cap P, C \cap P$ are mutually disjoint, and choose $Q \in \mathcal{M}$ with $Q \geq P$ so that $A \cap Q, B \cap Q, C \cap Q$ lie in the unbounded components of $A \cap P, B \cap P, C \cap P$ respectively. Let β be a generating loop in Q. Now, there is a unique subpath, $\alpha \subseteq \beta$, with $\alpha \cap A = \partial\alpha$ such that if we connect the endpoints of α by a path α' in $A \cap P$, then $\alpha \cup \alpha'$ is a generating loop in P. Let β_A be the complementary arc of α in β. (We can alternatively define β_A as the projection of the subpath $\tilde{\beta}[\mathrm{init}_{\tilde{\beta}}(\tilde{A}), \mathrm{final}_{\tilde{\beta}}(\tilde{A})] \subseteq \tilde{\beta} \subseteq \tilde{Q}$ to β.) We similarly define the subpaths β_B and β_C. Now, Lemma 10.2 tells us that the subpaths $\{\beta_A, \beta_B, \beta_C\}$ are "unnested" in the sense that none is contained in any other. Now, three unnested arcs in the circle have a cyclic order, namely the cyclic order of the three initial points, which is necessarily the same as the cyclic order of the three final points. This therefore determines the value of $\sigma(A, B, C)$.

We shall need a variation of this construction. Given any subsets, $A, B \subseteq X$, we write $A \sim B$ to mean that the Hausdorff distance from A to B is finite, i.e. there is some $r \geq 0$ such that $A \subseteq N(B, r)$ and $B \subseteq N(A, r)$. Note that this is an equivalence relation on the set of subsets of X.

Let's again suppose that $\mathcal{A}$ is a collection of connected one-ended subsets of X, with A separating in M_A, where M_A is the associated element of $\mathcal{M}$ for all $A \in \mathcal{A}$. If place of $(*)$, we assume:

$(**)$ If $A, B \in \mathcal{A}$ and $A \nsim B$, then for all $r \geq 0$, there is some $M \in \mathcal{M}$ with $\rho(A \cap M, B \cap M) \geq r$.

This time, we shall put a cyclic order on the quotient, $\mathcal{A}/\sim$.

In other words, we want to define a map, $\sigma : \{(A, B, C) \in \mathcal{A} \mid A \nsim B \nsim C \nsim A\} \longrightarrow \{0, 1\}$ which is a cyclic order on any transversal to $\sim$, and with the property that if $A \sim A' \nsim B \nsim C \nsim A$, then $\sigma(A, B, C) = \sigma(A', B, C)$. Now, we can define $\sigma(A, B, C) = \sigma_{\{A,B,C\}}(A, B, C)$ exactly as before. This gives a cyclic order on any transversal. We therefore need only verify the second property.

This is probably best seen using the second description of the cyclic order. Choose $P \geq M_A, M_{A'}, M_B, M_C$ so that $\rho(A \cap P, B \cap P), \rho(A \cap P, C \cap P), \rho(A' \cap P, B \cap P)$ and $\rho(A' \cap P, C \cap P)$, are all greater than r. Now choose a distant generating loop, β, and let $\beta_A, \beta_{A'}, \beta_B, \beta_C$ be the subpaths described above. Now we can assume that the initial point, x_A, of β_A is within a distance r along β_A from the initial point of $\beta_{A'}$, for otherwise, we could divert β by adjoining an arc in P of length at most r from the initial point of

A to some point of A', or from the initial point of A' to some point of A. Now the initial points, x_B and x_C of the subpaths β_B and β_C are each a distance greater than r from either x_A or $x_{A'}$. Thus the cyclic order of x_A, x_B, x_C on β is the same as that $x_{A'}, x_B, x_C$. Thus determines the cyclic order on $\sigma(A, B, C) = \sigma(A', B, C)$. This shows that σ gives a cyclic order on $\mathcal{A}/\sim$ as required.

As an example of this construction, we consider a set, $\mathcal{B}$, of directed (uniformly) uniform bi-infinite paths. Given $\alpha, \beta \in \mathcal{B}$, we write $\alpha \sim^+ \beta$ if $\alpha^+(x) \sim \beta^+(y)$ for some (hence every) $x \in \alpha$ and $y \in \beta$. We see that $\sim^+$ is an equivalence relation on $\mathcal{B}$. Suppose the set $\mathcal{B}$ satisfies condition $(**)$. We put a cyclic order on $\mathcal{B}/\sim^+$ as follows.

Given $\alpha \in \mathcal{B}$ choosing some $x \in \alpha$, we have a separating set $\Lambda_\alpha^+ = \Lambda^+(\alpha, x)$ in $M(\alpha, x)$ as described above. Now, each set Λ_α^+ is one-ended, and the collection $\mathcal{A} = \{\Lambda_\alpha^+ \mid \alpha \in \mathcal{B}\}$ also satisfies $(**)$. We therefore get a cyclic order on $\mathcal{A}/\sim$ which we can identify with $\mathcal{B}/\sim^+$.

We need to check that this order doesn't depend on the choice of basepoints, $x \in \alpha$. This is best done by observing that we can define the same order by considering the set of all pairs (α, x) with $x \in \alpha \in \mathcal{B}$ simultaneously. The quotient of $\{\Lambda^+(\alpha, x) \mid x \in \alpha \in \mathcal{B}\}$ by the relation of finite Hausdorff distance can again be identified with $\mathcal{B}/\sim^+$.

By the same argument, we also see that the cyclic order we have defined doesn't depend on the "radius" involved in the definition of the sets Λ^+. (Consider all radii simultaneously.) The cyclic order we have defined is therefore quite natural.

11. Parallel geodesics.

Let (X, ρ) be a planar 2-complex with integral winding numbers. The main objective of this and the next section will be to show that a group, Γ, acting properly discontinuously cocompactly on X must contain an element of infinite order. After passing to a subgroup of index at most 2, we may as well assume that Γ is orientation preserving. The argument proceeds by analysing the way that Γ displaces a given bi-infinite geodesic. We shall assume that any bi-infinite geodesic, α, is parameterised by arc-length, i.e. $\rho(\alpha(t), \alpha(u)) = |t - u|$ for all $t, u \in \mathbf{R}$. The main result on which the remainder of the argument rests is:

Proposition 11.1 : *There is some constant, $r_1 \geq 0$, and an increasing function, $k_0 : [0, \infty) \longrightarrow [0, \infty)$ such that the following holds. Suppose $g \in \Gamma$ has finite order n, and that $\mathrm{rot}(g) = 0$. Suppose that α is a bi-infinite geodesic in X. Let $d = D_g(\alpha(0)) = \rho(\alpha(0), g\alpha(0))$. Then, there exist $t, u \in [r_1, k_0(nd)]$ such that $\rho(\alpha(t), g\alpha(u)) \leq r_1$.*

Proof : We choose $r_1 \geq 0$ so that $N(\alpha, r_0) \subseteq \Lambda \subseteq N(\alpha, r_1/2)$, where Λ is the neighbourhood of α defined at the end of Section 5. The definition of the function k_0 will become apparent during the course of the proof.

Recall that $\Lambda^+ = \Lambda^+(\alpha, \alpha(0))$ is a separating set in $M = M(\alpha, \alpha(0)) \in \mathcal{M}$ as defined in Section 10. Note that if $x, y \in \Lambda^+$ then x and y can be connected by a path in $\Lambda^+ \cap N(\alpha(0), a - r_1)^C \cap N(\alpha(0), b + r_1)$, where $a = \min\{\rho(\alpha(0), x), \rho(\alpha(0), y)\}$ and $b = \max\{\rho(\alpha(0), x), \rho(\alpha(0), y)\}$. We can suppose that $\Lambda^+ \subseteq N(\alpha([r_1, \infty)), r_1)$.

Now choose $P \in \mathcal{M}(g)$ with $P \geq g^i M$ for all i. Choose $Q \in \mathcal{M}(g)$ with $Q \geq P$, and $N(Q, r_1) \subseteq P$. Note that $\operatorname{diam}(\bigcup_i g^i M^C)$ is bounded up to an additive constant by nd, where $d = D_g(\alpha(0))$. Thus, by the uniformity of these constructions (in particular, the one-endedness of X is described in Section 3), we see that we can construct Q so that $Q^C \subseteq N(\alpha(0), l_1)$, where l_1 is some function of nd. Note that $\Lambda^+ \cap Q$ lies in the unbounded component of $\Lambda^+ \cap P$.

Now let γ be a generating loop in Q. Again by the uniformity of the construction, we can assume that $\gamma \subseteq N(Q^C, l_2)$, where l_2 depends only on $\operatorname{diam}(Q^C)$, and hence ultimately on nd. Let $l_3 = 2(l_1 + l_2) + r_1$. Thus any pair of points in $\Lambda^+ \cap Q \cap N(Q^C, l_2)$ are connected by a path in $\Lambda^+ \cap P \cap N(Q^C, l_3)$. Now let $k = l_1 + l_3 + 2r_1$. Thus, $\Lambda^+ \cap N(Q^C, l_3) \subseteq N(\alpha([r_1, k]), r_1/2)$. By g-invariance, we have also $g\Lambda^+ \cap N(Q^C, l_3) \subseteq N(g\alpha([r_1, k]), r_1/2)$. We claim that there exist $t, u \in [r_1, k]$ with $\rho(\alpha(t), g\alpha(u)) \leq r_1$.

Suppose, for contradiction that $\rho(\alpha([r_1, k]), g\alpha([r_1, k])) > r_1$. It follows that $\Lambda^+ \cap g\Lambda^+ \cap N(Q^C, l_3) = \emptyset$.

In what follows we shall take indices mod n, noting that $g^n = 1$. Let $\mathcal{B} = \{g^i \Lambda^+ \mid i \in \mathbf{Z}_n\}$. We define $\mathcal{S}_Q = \mathcal{S}_Q(\mathcal{B})$ and $\mathcal{S}_P = \mathcal{S}_P(\mathcal{B})$ exactly as in Section 10. These are $\mathbf{Z}$-invariant collections of subsets of $\tilde{Q}$ and $\tilde{P}$ respectively, which project to sets of the form $g^i \Lambda^+ \cap Q$ or $g^i \Lambda^+ \cap P$. As before, we have arranged that if $S \in \mathcal{S}_Q$, then S lies in the unbounded component of some $S_P \in \mathcal{S}_P$. Let β be the lift of the generating loop γ to $\tilde{Q}$.

Now since $\operatorname{rot}(g) = 0$, there is a lift, h, of g to $\tilde{P}$ with $h^n = 1$. Now, given $i \in \mathbf{Z}_n$, we define an order, $<_i = <_{h^i \beta}$ on $\mathcal{S}_Q \equiv \mathcal{S}_P$ as in Section 10, namely, we write $R <_i S$ to mean that $\operatorname{init}_{h^i \beta}(R) < \operatorname{init}_{h^i \beta}(S)$. This is a partial order on $\mathcal{S}_Q$. (It's conceivable that distinct elements of $\mathcal{S}_Q$ might have identical initial points on $h^i \beta$. We're not assuming a-priori that these sets are disjoint.)

Now choose any $S \in \mathcal{S}_Q$ which projects to $\Lambda^+ \cap Q$. We have arranged that $S \cap hS \cap \beta = \emptyset$, and so either $S <_0 hS$ or $hS <_0 S$. Without loss of generality, we can suppose that $S <_0 hS$.

We now claim that $S <_i hS$ for all $i \in \mathbf{Z}_n$. The argument is essentially the same as that of Lemmas 10.2 and 10.3 combined. We know that we can connect any pair of points in $\beta \cap S$ and $h^i \beta \cap S$ by a path ϵ in S_P. Moreover, we can choose ϵ so that it projects to a path in $\Lambda^+ \cap P \cap N(Q^C, l_3)$. But we have arranged that $\Lambda^+ \cap g\Lambda^+ \cap N(Q^C, l_3) = \emptyset$. Thus, $\epsilon \cap hS_P = \emptyset$. Similarly, we can connect any two points of $\beta \cap S$ and $h^i \beta \cap S$ by a path in hS_P which does not meet S_P. These facts are all we need to make the arguments of Lemmas 10.2 and 10.3 work to show that $S <_i hS$ as claimed.

Now, by the g-invariance of the constructions, we see that $h^j S <_{i+j} h^{j+1} S$ for all $i, j \in \mathbf{Z}_n$. Putting $i = -j$, we get that $h^j S <_0 h^{j+1} S$ for all j. By transitivity of $<_0$, we conclude inductively that $S <_0 h^j S$, so we get the contradiction $S <_0 h^n S = S$.

We have contradicted the assumption that $\rho(\alpha([r_1, k], g\alpha([r_1, k])) > r_1$. In other words there exist $t, u \in [r_1, k]$ with $\rho(\alpha(t), g\alpha(u)) \leq r_1$. Here r_1 is universal, and k depends only on nd. $\diamond$

Clearly, by translating the parameterisation of α, we see that for any $s \in \mathbf{R}$, there exist $t, u \geq s$ with $\rho(\alpha(t), g\alpha(u)) \leq r$. By replacing α by $-\alpha$, we see likewise that there exist $t', u' \leq s$ with $\rho(\alpha(t'), g\alpha(u')) \leq r_1$. We conclude:

Corollary 11.2 : *With the hypotheses of Proposition 11.2, we can find bi-infinite sequences, $(t_i)_{i \in \mathbf{Z}}$ and $(u_i)_{i \in \mathbf{Z}}$, with $t_i, u_i \to \infty$ as $i \to \infty$ and $t_i, u_i \to -\infty$ as $i \to -\infty$, with $\rho(\alpha(t_i), g\alpha(u_i)) \le r_1$ for all $i \in \mathbf{Z}$.* $\diamond$

In fact, we can say more. First some definitions.

Definition : Suppose α, β are bi-infinite geodesics. We say that α and β are *r-parallel* if $\rho(\alpha(t), \beta(t)) \le r$ for all $t \in \mathbf{R}$. We say that α, β are *parallel*, and write $\alpha \parallel \beta$, if they are *r*-parallel for some $r \ge 0$.

Clearly parallelism is an equivalence relation on geodesics.

Proposition 11.3 : *Suppose that $g \in \Gamma$ has finite order, n, and that $\mathrm{rot}(g) = 0$. Suppose α is a bi-infinite geodesic in X. Then, α and $g\alpha$ are k-parallel, where k is bounded above by some function of $nD_g(\alpha(0))$ (and hence of $nD_g(\alpha(t))$ for any $t \in \mathbf{R}$).*

Proof : This follows directly from the fact that the points, t_i and u_i in Corollary 11.2 can be chosen so the gaps, $|t_{i+1} - t_i|$ and $|u_{i+1} - u_i|$ are all bounded above by some function of $nD_g(\alpha(0))$. This follows inductively from Proposition 11.1, and the following observation (Lemma 11.4) which ensures that the displacements $D_g(\alpha(t_i))$ are uniformly bounded in terms of the initial displacement, $D_g(\alpha(0))$. $\diamond$

Lemma 11.4 : *Suppose α, β are geodesics with $\rho(\beta(t), \beta(u)) \le r$. Then $\rho(\alpha(t), \beta(t)) \le \rho(\alpha(0), \beta(0)) + 2r$.*

Proof : $\rho(\alpha(t), \beta(t)) \le \rho(\alpha(t), \beta(u)) + \rho(\beta(t), \beta(u)) \le r + |t - u| = r + |\rho(\alpha(0), \alpha(t)) - \rho(\beta(0), \beta(u))| \le r + \rho(\alpha(0), \beta(0)) + \rho(\alpha(t), \beta(u)) \le 2r + \rho(\alpha(0), \beta(0))$ $\diamond$

Now, Lemma 9.2 tells us that if $g \in \Gamma$ has finite order and $\mathrm{rot}(g) \ne 0$, then for any bi-infinite geodesic, α, we have $D_g(\alpha(t)) \to \infty$ as $t \to \infty$ and as $t \to -\infty$. Thus, we see:

Corollary 11.5 : *If $g \in \Gamma$ is of finite order and α is any bi-infinite geodesic in X, then $\alpha \parallel g\alpha$ if and only if $\mathrm{rot}(g) = 0$.* $\diamond$

Let us now assume that Γ is a torsion group. We eventually aim to derive the contradiction that Γ is finite, which we shall get to in Section 12. We can begin by drawing some immediate conclusions.

Let $\Gamma_0 = \{\gamma \in \Gamma \mid \mathrm{rot}(g) = 0\}$.

Lemma 11.6 : Γ_0 *is a normal subgroup of Γ.*

Proof : The fact that Γ_0 is a subgroup follows from Lemma 6.1 and Corollary 11.5. The fact that it is normal follows from the conjugacy invariance of rotational part. $\diamond$

Now let's choose a bi-infinite geodesic (using Lemma 6.1), and let $\mathcal{B}$ be the set of Γ-images. Let $\mathcal{T} = \mathcal{B}/\parallel$ be the set of parallel classes. Thus, Γ/Γ_0 acts on $\mathcal{T}$. By Lemma 9.2,

we see that this action is effective, and that if $\alpha, \beta \in \mathcal{B}$ belong to distinct parallel classes, then $\rho(\alpha(t), \beta(t)) \to \infty$ as $t \to \infty$ and as $t \to -\infty$. Thus, the relation $\parallel$ agrees with the relation $\sim^+$ defined in Section 10. Moreover, the condition $(**)$ is satisfied, so that $\mathcal{T} = \mathcal{B}/\parallel = \mathcal{B}/\sim^+$ admits a canonical cyclic order. Since Γ is orientation preserving, Γ/Γ_0 must preserve this order. By Corollary 8.2, Γ/Γ_0 is locally cyclic. But Γ acts cocompactly on X. Thus Γ and hence Γ/Γ_0 is finitely generated. It follows that Γ/Γ_0 is finite cyclic. In particular, we conclude:

Proposition 11.7 : Γ_0 *has finite index in* Γ. $\diamond$

Thus, Γ_0 itself acts cocompactly on X, and is therefore also finitely generated. Since X is one ended, so is Γ_0.

Thus, for the purposes of deriving a contradiction in Section 12, we can assume that every element of Γ has zero rotational part.

12. Displacement of geodesics.

In this section we aim to prove one of the central results of this paper, namely Theorem 12.9. To this end, we shall need to define the "displacement" of geodesics. As in Section 11, let (X, ρ) be a planar 2-complex, with integral winding numbers.

Suppose that α, β are geodesics and that $t, u, t_0, u_0 \in \mathbf{R}$ with $t \geq t_0$ and $u \geq u_0$. Now,

$$
\begin{aligned}
|(t - u) - (t_0 - u_0)| &= |(t - t_0) - (u - u_0)| \\
&= |\rho(\alpha(t), \alpha(t_0)) - \rho(\beta(u), \beta(u_0))| \\
&\leq \rho(\alpha(t), \beta(u)) + \rho(\alpha(t_0), \beta(u_0)).
\end{aligned}
$$

Now if $t' \geq t_0$ and $u' \geq u_0$, then

$$
\begin{aligned}
|(t - u) - (t' - u')| &\leq |(t - u) - (t_0 - u_0)| + |(t' - u') - (t_0 - u_0)| \\
&\leq \rho(\alpha(t), \beta(u)) + \rho(\alpha(t'), \beta(u')) + 2\rho(\alpha(t_0), \beta(u_0)).
\end{aligned}
$$

Definition : We say that bi-infinite geodesics, α, β are *r-close* if there exist bi-infinite sequences, $(t_i)_{i \in \mathbf{Z}}$ and $(u_i)_{i \in \mathbf{Z}}$ with $t_i \to \infty$ and $u_i \to \infty$ as $i \to \infty$ and $t_i \to -\infty$ and $u_i \to -\infty$ as $i \to -\infty$, so that $\rho(\alpha(t_i), \beta(u_i)) \leq r$ for all $i \in \mathbf{Z}$.

From the above calculation, choosing t_0 and u_0 sufficiently negative, we see that if $t, u, t', u' \in \mathbf{R}$ with $\rho(\alpha(t), \beta(u)) \leq r$ and $\rho(\alpha(t'), \beta(u')) \leq r$, then $|(t - u) - (t' - u')| \leq 4r$. Thus, the quantity, $t - u$, is well defined up to the additive constant $4r$. For definiteness, let's define $\Delta(\alpha, \beta) = \Delta_r(\alpha, \beta) = \sup\{t - u \mid \rho(\alpha(t), \beta(u)) \leq r\}$. Thus, for any t, u with $\rho(\alpha(t), \beta(u)) \leq r$ we have $|(t - u) - \Delta(\alpha, \beta)| \leq 4r$. In fact:

Lemma 12.1 :   *If* α, β *are r-close, and* $t, u \in \mathbf{R}$, *then* $|(t - u) - \Delta(\alpha, \beta)| \leq 3r + \rho(\alpha(t), \beta(u))$.

Proof : Choose $t', u' \in \mathbf{R}$ with $\rho(\alpha(t'), \beta(u')) \leq r$ and with $t' - u'$ arbitrarily close to $\Delta(\alpha, \beta)$, and apply the above inequality using any sufficiently small t_0 and u_0. $\diamondsuit$

Lemma 12.2 : If α, β are r-close, then for some $t \in \mathbf{R}$, we have $\rho(\alpha(t), \beta(t)) \leq 5r + |\Delta(\alpha, \beta)|$.

Proof : Choose any $t, u \in \mathbf{R}$ with $\rho(\alpha(t), \beta(u)) \leq r$. Then $\rho(\alpha(t), \beta(t)) \leq \rho(\alpha(t), \beta(u)) + \rho(\beta(t), \beta(u)) \leq r + |t - u| \leq 5r + |\Delta(\alpha, \beta)|$. $\diamondsuit$

Of course, we do not expect closeness to be an equivalence relation, unlike the notion of parallelism defined in Section 11. However, we note:

Lemma 12.3 : Suppose α, β, γ are bi-infinite geodesics, and that α is r-close to both β and γ. Suppose that β and γ are k-parallel, then $|\Delta(\alpha, \beta) - \Delta(\alpha, \gamma)| \leq 8r + k$.

Proof : Choose $t, u \in \mathbf{R}$ with $\rho(\alpha(t), \beta(u)) \leq r$. Thus, $|(t - u) - \Delta(\alpha, \beta)| \leq 4r$ and, applying Lemma 12.2, we have

$$
\begin{aligned}
|(t - u) - \Delta(\alpha, \gamma)| &\leq 3r + \rho(\alpha(t), \gamma(u)) \\
&\leq 3r + \rho(\alpha(t), \beta(u)) + \rho(\beta(u), \gamma(u)) \\
&\leq 3r + (r + k) = 4r + k.
\end{aligned}
$$

$\diamondsuit$

Now suppose that Γ is a torsion group acting properly discontinuously cocompactly on X. Let's suppose, in addition, that Γ is orientation preserving and that every element has zero rotational part.

Let α be any bi-infinite geodesic in X (using Lemma 6.1). By Corollary 11.2, if $g \in \Gamma$, then $g\alpha$ is r_1-close to α, for some fixed constant r_1 (independent of g). Set $\Delta(g) = \Delta_{r_1}(\alpha, g\alpha)$. This gives us a map, $\Delta : \Gamma \longrightarrow \mathbf{R}$.

Lemma 12.4 : If $g \in \Gamma$, then $|\Delta(g) + \Delta(g^{-1})| \leq 8r_1$.

Proof : Choose $t, u \in \mathbf{R}$ with $\rho(\alpha(t), g\alpha(u)) \leq r_1$. Thus, $\rho(\alpha(u), g^{-1}\alpha(t)) \leq r_1$, and so $|\Delta(g) - (t - u)| \leq 4r_1$ and $|\Delta(g^{-1}) - (u - t)| \leq 4r_1$, and the result follows. $\diamondsuit$

Since Γ acts properly discontinuously cocompactly on X, it must be finitely generated. Let $A \subseteq \Gamma$ be a finite symmetric generating set. Now, by Proposition 11.3, $\alpha \parallel g\alpha$ for all $g \in \Gamma$. In particular, there is some $r_2 \geq 0$ such that for all $h \in A$, α is r_2-parallel to $h\alpha$.

Lemma 12.5 : If $g \in \Gamma$ and $h \in A$, then $|\Delta(g) - \Delta(gh)| \leq 8r_1 + r_2$.

Proof : Now, $h\alpha$ is r_2-parallel to α. Thus, $gh\alpha$ is r_2-parallel to $g\alpha$. By Corollary 11.2, $g\alpha$ and $gh\alpha$ are both r_1-close to α. Thus, by Lemma 12.3, we have $|\Delta(g) - \Delta(gh)| = |\Delta_{r_1}(\alpha, g\alpha) - \Delta_{r_1}(\alpha, gh\alpha)| \leq 8r_1 + r_2$. $\diamond$

Let $r_3 = 8r_1 + r_2$. Let $B = \{g \in \Gamma \mid |\Delta(g)| \leq r_3\}$.

Lemma 12.6 : *B is infinite.*

Proof : Let Y be the Cayley graph of Γ corresponding to the generating set, A. It has vertex set Γ, and $g, h \in \Gamma$ are adjacent if and only if $g^{-1}h \in A$. Lemma 12.5 tells us that if g, h are adjacent, then $|\Delta(g) - \Delta(h)| \leq r_3$. Now, Γ and hence Y is one-ended. Thus, if $B \subseteq \Gamma$ were finite, $Y \setminus B$ would have precisely one unbounded component. The vertex set of this component must map under Δ into either $[r_3, \infty)$ or $(-\infty, -r_3]$. Let's assume the former. Then $\{g \in \Gamma \mid g \leq -r_3\}$ is finite. But this is seen to contradict Lemma 12.4. We conclude that B must be infinite as claimed. $\diamond$

We now proceed to show that B is finite, thereby deriving a contradiction.

Suppose that $g \in B$. By Lemma 12.2, there is some $t \in \mathbf{R}$ such that $D_g(\alpha(t)) = \rho(\alpha(t), g\alpha(t)) \leq 5r_1 + |\Delta(g)| \leq 5r_1 + r_3$. Now, since Γ acts cocompactly on X, some Γ-image of $\alpha(t)$ must lie a bounded distance from any given point of X, say $\alpha(0)$. Thus, some Γ-conjugate of g moves the point $\alpha(0)$ a bounded distance. Since X is locally finite, there are only finitely many possibilities for such a conjugate. In other words, B lies in a finite union of conjugacy classes in Γ. In particular, we conclude:

Lemma 12.7 : *There is some $n \in \mathbf{N}$ such that $g^n = 1$ for all $g \in B$.* $\diamond$

Now, by Proposition 11.3, given any $g \in B$ and $t \in \mathbf{R}$, α is k-parallel to $g\alpha$, where k is bounded by some function of $nD_g(\alpha(t))$. By Lemma 12.2, as above, we can choose this t so that $D_g(\alpha(t)) \leq 5r_1 + r_3$. It follows that k is, in fact, independent of $g \in B$. In particular, $D_g(\alpha(0)) \leq k$ for all $g \in B$. Now, since X is locally finite, we conclude:

Lemma 12.8 : *B is finite.* $\diamond$

This is somewhat at odds with Lemma 12.6, so we are forced to admit that Γ cannot be a torsion group after all.

Now, it was shown at the end of Section 11 that any torsion group acting properly discontinuously cocompactly on X has a finite index subgroup with zero rotational parts. We have thus shown:

Theorem 12.9 : *Suppose Γ acts properly discontinuously cocompactly on a planar 2-complex with integral winding numbers. Then Γ contains an element of infinite order.* $\diamond$

13. Virtual Surface groups.

Suppose (X, ρ) is a planar 2-complex with integral winding numbers. Suppose Γ acts properly discontinuously cocompactly on X. We have seen (Theorem 12.9) that Γ contains an element of infinite order. (We do not need to assume for the moment that Γ is orientation preserving.)

If $G \leq \Gamma$ is a subgroup, we say that G has *strict codimension-one* if Y/G has more than one end, where Y is some Cayley graph of Γ. One can show that this is independent of the choice of Y. Indeed one could take Y to be any space on which Γ acts properly discontinuously cocompactly (for example X). We say that G has *codimension-one* if some finite index subgroup of G has strict codimension-one. (This is a slight variant on the terminology of [DuSw] — they use "codimension-one" for what we have called "strict codimension-one".)

Proposition 13.1 : *Any infinite cyclic subgroup of Γ has codimension-one.*

Proof : Suppose $g \in \Gamma$. Let β be any $\langle g \rangle$-invariant path (so that $\beta/\langle g \rangle$ is compact). Such a path is necessarily uniform, so we can construct the sets $\Lambda = \Lambda(\alpha, r)$, $C_L = C_L(\alpha, r)$ and $C_R = C_R(\alpha, r)$ as defined in Section 4. These sets are all $\langle g^2 \rangle$-invariant, and $\Lambda/\langle g^2 \rangle$ is compact. Writing $X/\langle g^2 \rangle$ as a union of $\Lambda/\langle g^2 \rangle$, $C_L/\langle g^2 \rangle$ and $C_R/\langle g^2 \rangle$, and noting that the latter two sets are disjoint and unbounded, we see that $X/\langle g^2 \rangle$ has more than one end. $\diamondsuit$

(In fact, by taking an increasing sequence of radii, r, in the above argument, we see that $X/\langle g^2 \rangle$ has precisely two ends.)

In summary, we we have shown:

Proposition 13.2 : *Γ is finitely generated, one-ended, contains an element of infinite order, and every infinite cyclic subgroup has codimension-one.* $\diamondsuit$

Now it is shown in [Bo] that a group with the properties described by Proposition 13.2 has to be planar. In the case of almost finitely presented groups (i.e. FP_2 over $\mathbf{Z}_2$) this can be deduced from the results and methods of [DuSa] and [DuSw]. Moreover, Graham Niblo has suggested how these arguments might be adapted to deal with the general case.

In the present situation, we have some additional information, namely the winding number which gives us directly the cyclic orders which feature in [Bo]. In the remainder of this section we sketch a direct argument to complete the proof, referring to [Bo] for details.

Our argument makes use of ideas from [DuSw] and [Sw]. (Instead of the "tracks" of [DuSw], we will speak in terms of "periodic paths". The separation properties of these tracks are expressed in terms of the planarity of the complex X.)

We thus aim to show:

Theorem 13.3 : *Suppose a group, Γ, acts properly discontinuously cocompactly on a planar 2-complex with integral winding numbers. Then Γ is a virtual surface group.*

The essential facts we will need about Γ will be that it satisfies the conclusion of

Proposition 13.2, together with certain cyclic order properties of ends of two-ended sub-groups.

As usual, one of the key steps will be in separating the "euclidean" case (where Γ is virtually $\mathbf{Z} \oplus \mathbf{Z}$) from the "hyperbolic" case (where Γ is fuchsian). Unlike the approach of Kleiner, we won't make explicit use of hyperbolicity. However, as with all approaches to date, we shall deal with the fuchsian case by appealing to the result of Tukia, Gabai and Casson and Jungreis:

Theorem 13.4 : [T,Ga,CJ] *If a group, Γ, acts as a convergence group on the circle, S^1, then it also admits a properly discontinuous isometric action on the hyperbolic plane (such that the induced action on the boundary is topologically conjugate to the original).*
$$\diamond$$

Thus, if Γ is finitely generated and one-ended, then it is a virtual surface group as required.

We shall recognise convergence actions on the circle via a result of Swenson [Sw]. Let $\Phi(S^1)$ be the space of distinct unordered pairs in S^1. (Thus, $\Phi(S^1)$ is topologically an open Möbius band.) Suppose that G is a two-ended (i.e. virtually cyclic) group acting on S^1. We say that G is *loxodromic* if it acts as a convergence group with limit set consisting of a pair of distinct ponts, $\{x, y\}$. (In other words, there is some $g \in G$ with $g^n | S^1 \setminus \{x\}$ converging to y, and $g^{-n} | S^1 \setminus \{y\}$ converging locally uniformly to x.)

The following is a simple consequence of the main result of [Sw]:

Theorem 13.5 : *Suppose $\Pi \subseteq \Phi(S^1)$ is a discrete Γ-invariant with $\lambda \cap \mu = \emptyset$ for distinct $\lambda, \mu \in \Pi$, and with Π/Γ finite. Suppose that the stabiliser of each element $\lambda \in \Pi$ is two-ended and loxodromic (with limit set λ). Suppose also that every pair of points in $S^1 \setminus \bigcup \Pi$ are separated by an element of Π. Then Γ acts as a convergence group on S^1.*

Proof : The hypothesis on the separation property of Π is just a convenient way of expressing the fact that $\bigcup \Pi$ is dense in S^1, and that Π is "cross-connected" as described below. The result thus follows from [Sw].
$$\diamond$$

We shall in turn recognise this property by using cyclic orders on ends of periodic paths in X. (A similar idea was used by Scott in his proof of the Torus Theorem for 3-manifolds.) We begin by introducing some general terminology and notation relating to cyclic orders. (More general constructions of this type are discussed in [Bo]).

Let (E, σ) be a cyclically ordered set. Let $\Phi(E) = \{\{x, y\} \mid x, y \in E, x \neq y\}$ be the set of "pairs" in E. Given two disjoint pairs, $\lambda = \{x, y\}$ and $\mu = \{z, w\}$, they either *cross* (so that $\{x, y\}$ separates z from w in the cyclic order) in which case we write $\lambda \times \mu$ or $x - \mu - y$ or $z - \lambda - w$, or else the don't cross, in which case, we write $\lambda : \mu$. Given $A, B \subseteq E$ and $\lambda \in \Phi(E)$, we write $A - \lambda - B$ to mean that $x - \lambda - y$ for all $x \in A$ and all $y \in B$.

Now let $\Phi(E) = \{\{x, y\} \mid x, y \in E, x \neq y\}$. A *pattern* on E is a subset $\Pi \subseteq \Phi(E)$ such that $\lambda \cap \mu = \emptyset$ for all distinct $\lambda, \mu \in \Pi$. We say that Π is *full* if $\bigcup E = \Pi$. We define the following three finiteness conditions on a pattern, Π:

(F0): If $x, y, z, w \in E$ are distinct, then $\{\nu \in \Pi \mid \{x, y\} - \nu - \{z, w\}\}$ is finite.

(F1): If $\lambda, \mu \in E$ with $\lambda : \mu$, then $\{\nu \in \Pi \mid \lambda - \nu - \mu\}$ is finite.

(F2): If $\lambda, \mu \in E$ with $\lambda \neq \mu$, then $\{\nu \in \Pi \mid \nu \times \lambda, \nu \times \mu\}$ is finite.

The following is easily verified:

Lemma 13.6 : *If Π is full, then (F1) and (F2) implies (F0).* $\diamondsuit$

We say that a full pattern is *discrete* if it satisfied (F0) (or equivalently (F1) and (F2)).

If Π is a pattern in S^1, then (F0) is equivalent to Π being discrete in $\Phi(S^1)$. If $\bigcup \Pi$ is dense in S^1, then if Π satisfies (F0) in $\bigcup \Pi$, then it satisfies (F0) also in S^1.

Following [Sw], we say that a pattern, Π, is *cross-connected* if given any $\lambda, \mu \in \Pi$, there is a finite sequence, $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_n = \mu$, of elements of Π with $\lambda_i \times \lambda_{i+1}$ for all i. Any pattern can be decomposed into cross-connected components. Moreover, it is not hard to see [Bo]:

Lemma 13.7 : *If Π is a discrete pattern, then the set, $\mathcal{P}$, of cross-connected components of Π can be canonically embedded in the vertex set of a simplicial tree, Σ, in such a way that if $P, Q, R \in \mathcal{P}$ then P separates Q from R in Σ if and only if there is some $\lambda \in P$ such that $Q - \lambda - R$.* $\diamondsuit$

We now introduce group actions. Suppose that E is a cyclically ordered set, and that Γ acts on E by order-preserving maps. Suppose that Π is a Γ-invariant pattern. Given $\lambda \in \Pi$, we write $\Gamma(\lambda)$ for the (setwise) stabiliser of λ. We say that Π is a Γ-*pattern* if Π/Γ is finite and $\Gamma(\lambda)$ is two-ended for all $\lambda \in \Pi$.

Lemma 13.8 : *Suppose Π is a discrete full Γ-pattern on E. Suppose that for all $\lambda \in \Pi$ there is some $\mu \in \Pi$ with $\mu \times \lambda$ and some $g \in \Gamma(\lambda)$ with $g\mu : \mu$. Then, E is a dense cyclic order.*

Proof : Let $x \in E$. Given $y, z \in E$, we show that there is some $\nu \in \Pi$ with $x - \nu - \{y, z\}$. Let λ be the element of Π containing x, and let μ and g be as in the hypotheses. Without loss of generality, we have $\mu - g\mu - x$. We must have $y - g^n\mu - x$ for all sufficiently large n, otherwise $\{n \in \mathbf{N} \mid \mu - g^n\mu - \{x, y\}\}$ would be infinite, contradicting discreteness. Similarly $z - g^n\mu - x$ for all sufficiently large n. We thus set $\nu = g^n\mu$ for large enough n. $\diamondsuit$

It follows that if E is countable then it admits a conical completion to a circle, S^1. The action of Γ extends to an action by homeomorphism on S^1. A simple extension the argument of Lemma 13.8 shows:

Lemma 13.9 : *With the hypotheses of Lemma 13.8, if E is countable, then in the induced action of Γ on S^1, $\Gamma(\lambda)$ is a loxodromic convergence group for all $\lambda \in \Pi$.* $\diamondsuit$

Thus, if Π is cross-connected, then it follows by Theorem 13.5 that Γ acts as a convergence group on S^1. If Γ is finitely generated and one-ended, then it is a virtual surface group.

We now move on to more geometric considerations. Suppose Γ is a group, and that $g, h \in \Gamma$ have infinite order. We say that g and h are *commensurate* if there exist $m, n \in \mathbf{Z} \setminus \{0\}$ with $g^m = h^n$. We define the *commensurator*, $\mathrm{Comm}(g)$, of g to be the set of $h \in \Gamma$ such that hgh^{-1} is commensurate with g. We see that $\mathrm{Comm}(g)$ is a subgroup of Γ containing g.

Suppose, now, that Γ acts properly discontinuously on a metric complex, (X, ρ). We say that a bi-infinite path, $\alpha \subseteq X$, is *periodic* if there is some infinite order $g \in \Gamma$ such that $g\alpha = \alpha$. It follows that $\alpha/\langle g \rangle$ is compact, and that α is a uniform path. We refer to g as a *period* of α, and to α as an *axis* of g. Clearly, two periods of the same axis will be commensurate, and every infinite order element has an axis.

Definition : We say that two uniform paths α and β are *parallel* if they are at finite Hausdorff distance, i.e. there is some $r \geq 0$ such that $\alpha \subseteq N(\beta, r)$ and $\beta \subseteq N(\alpha, r)$.

Note that unlike the definition of "parallel" for geodesics in Section 11, we are not taking account of parameterisations or of direction. If $\vec{\alpha}$ and $\vec{\beta}$ are directed, we say they are *consistently directed* if the positive rays are at a finite Hausdorff distance and the negative rays are at finite Hausdorff distance.

Definition : We say that two uniform paths α and β are *divergent* if for all $r \geq 0$, there is some compact $K \subseteq X$ such that $\rho(\alpha \cap K^C, \beta \cap K^C) \geq r$.

The following is easily verified:

Lemma 13.10 : *If α and β are periodic paths with periods g and h, then either α and β are parallel and g and h are commensurate, or else α and β are divergent and g and h are not commensurate.* $\diamondsuit$

Now, given $g \in \Gamma$, define $D(g) = \inf\{x \in X \mid \rho(x, gx)\}$. Clearly, D is a conjugacy invariant, and $D(g^n) \leq nD(g)$ for all $n \geq 0$. It's easily seen that some (hence every) axis of g is quasigeodesic if and only if $D(g^n)$ is bounded below by some increasing linear function of n. If Γ acts cocompactly on X, then this is the same as saying that $\langle g \rangle$ is quasiisometrically embedded in Γ. If $h \in \mathrm{Comm}(g)$, then it's easily seen that h^2 commutes with g^n for some n. Thus, either $\mathrm{Comm}(g)$ is two-ended or it contains an isometric copy of $\mathbf{Z} \oplus \mathbf{Z}$.

Suppose that $\alpha \subseteq X$ is a directed uniform path. We set $\Lambda(\alpha) = \Lambda(\alpha, r_0)$ for fixed sufficiently large r_0, as described in Section 5. Note that $\Lambda(\alpha)$ lies inside a uniform neighbourhood of α. Let $C_L(\alpha) = C_L(\alpha, r_0)$ and $C_R(\alpha) = C_R(\alpha, r_0)$ be the left and right deep complements. (This construction is only really needed where α is periodic with some period, g. The important facts are that all three sets are connected and $\langle g \rangle$-invariant, that $C_L(\alpha) \cap C_R(\alpha) = \emptyset$ and that $\Lambda(\alpha)/\langle g \rangle$ is compact, and $C_L(\alpha)/\langle g \rangle$ and $C_R(\alpha)/\langle g \rangle$ are unbounded.)

Suppose that $\vec{\alpha}$ and $\vec{\beta}$ are directed uniform paths, and that $\Lambda(\alpha) \cap \Lambda(\beta)$ is compact. By the constructions of Section 10, we see that there is a well-defined cyclic order on $\{\vec{\alpha}, \vec{\beta}, -\vec{\alpha}, -\vec{\beta}\}$, which is better thought of as a cyclic order on the set of positive and

negative rays or "ends" of α and β, i.e. $\{\alpha^+, \beta^+, \alpha^-, \beta^-\}$. We say that α and β *cross* if $\{\alpha^-, \alpha^+\}$ crosses $\{\beta^-, \beta^+\}$ in this cyclic order, and we write $\alpha \times \beta$. Otherwise, we write $\alpha : \beta$. In the latter case, $\vec{\alpha}$ and $\vec{\beta}$ may be consistently or oppositely directed, depending on the cyclic order on their endpoints. Note that $\Lambda(\alpha) \cap C_L(\beta)$ consists of 0,1 or 2 ends of $\Lambda(\alpha)$ (up to a compact set), depending on the above cyclic order. We shall say that α and β are *strictly disjoint* if $\Lambda(\alpha) \cap \Lambda(\beta) = \emptyset$. This clearly implies $\alpha : \beta$.

Suppose again that Γ acts properly discontinuously on (X, ρ). Suppose that α is a uniform path, and that $h \in \Gamma$ with $h\alpha$ strictly disjoint from α, and such that α and $h\alpha$ are consistently directed. Then it's easy to see that h must have infinite order, and any axis, β, of h must cross α. If $x \in X$, then any path connecting x to $h^n x$ must cross $n - 1$ disjoint images of $\Lambda(\alpha)$. We see that $D(h^n)$ is bounded below by a linear function of n. Thus β is quasigeodesic.

Suppose that $\mathcal{A}$ is a Γ-invariant collection of periodic arcs in X, with $\mathcal{A}/\Gamma$ finite. Thus, $\mathcal{A}$ is locally finite in X. Note that, up to the action of Γ, there are only finitely many non-empty sets of the form $\Lambda(\alpha) \cap \Lambda(\beta)$ for $\alpha, \beta \in \mathcal{A}$. There is therefore a uniform bound on the diameters of those which are compact. In this way we can imagine the "crossings" of the elements of $\mathcal{A}$ as being local in X. Note that if $\Lambda(\alpha) \cap \Lambda(\beta)$ is compact and non-empty, then α and β must diverge.

Suppose that $\alpha \in \mathcal{A}$ with period g. We see that $\{\beta \in \mathcal{A} \mid \Lambda(\alpha) \cap \Lambda(\beta) \neq \emptyset\}/\langle g \rangle$ is finite. If $\beta : \alpha$ with $\beta \cap C_L(\alpha)$ compact, then there is a bound on how deeply β can enter $C_L(\alpha)$. Using these facts, we see that if $h \in \Gamma$ with $h\alpha : \alpha$ and $h\alpha$ and α consistently directed, then either $h\alpha$ is strictly disjoint from α or else α and $h\alpha$ diverge and $h^n\alpha$ is strictly disjoint from α for some $n \geq 0$. Either way, we see that h has infinite order, and any axis of h is quasigeodesic and crosses α.

Now suppose that Γ acts properly discontinuously cocompactly on X. We can also suppose (passing to a subgroup of index at most 2) that Γ is orientation preserving (in the sense that it preserves winding numbers, and hence also the cyclic orders defined earlier). The following lemma will get us started.

Lemma 13.11 : *Suppose α is a periodic arc in X. Then there is some periodic arc β which crosses α (so that α and β are divergent).*

Proof : Let g be a period of α, and let $\mathcal{A}$ be the set of Γ-images of α. We can suppose that no two elements of $\mathcal{A}$ cross — otherwise we are done. Now since $\mathcal{A}$ is locally finite, we can find some $h \in \Gamma$ with $\rho(\alpha, h\alpha) \geq r$ for arbitrarily large r. In particular, we can assume, without loss of generality, that $\Lambda(h\alpha) \subseteq C_R(\alpha)$. If α and $h\alpha$ are consistently directed, then h has infinite order, and its axis crosses α. If not, we need to try harder. Now, $C_L(\alpha)/\langle g \rangle$ is unbounded. Since the action of Γ on X is compact, we can find some $k \in \Gamma$ with $k\alpha \cap C_L(\alpha) \cap N(\alpha, s)^C \neq \emptyset$ for s arbitrarily large. By the compactness of $\Lambda(\alpha)/\langle g \rangle$, we can assume that $\Lambda(k\alpha)$ meets $\Lambda(\alpha)$ in a (possibly empty) compact set. Now, $k\alpha$ cannot cross α, so we must have $k\alpha : \alpha$. Also choosing r sufficiently large, we can suppose that $k\alpha$ and $h\alpha$ are strictly disjoint. Also α separates (in the sense of cyclic order) $h\alpha$ from $k\alpha$. In other words, the paths $\{\alpha, h\alpha, k\alpha\}$ are nested, and so some pair of them must be consistently directed. From the previous discussion, we see that at least one of h,

k or $h^{-1}k$ must be of infinite order, with its axis crossing α. $\diamondsuit$

Now let's suppose that $\mathcal{A}$ is a Γ-invariant collection of periodic arcs with $\mathcal{A}/\Gamma$ finite. As already observed, any two elements of $\mathcal{A}$ are either parallel or divergent. Two elements that cross are necessarily divergent. The following lemma is essentially the same as Lemma 4.5 of [DuSw]:

Lemma 13.12 : *Suppose $\alpha \in \mathcal{A}$ with period g. If $\beta \in \mathcal{A}$ crosses α then $g\beta$ does not cross β.*

Proof : Suppose $g\beta$ does cross β. By induction, $g^n\beta$ crosses β for all $n > 0$. Also $g^{n+1}\beta$ crosses $g^n\beta$ within a bounded distance of α. Beyond this, the paths $g^{n+1}\beta$ and $g^n\beta$ diverge. By considering the order of the crossings of $g^n\beta$ along β, we derive a contradiction (cf. [DuSw] or [Bo]). $\diamondsuit$

Note that β and $g\beta$ are necessarily consistently directed. Since g is not commensurate with the period of β, we see that for sufficiently large n, $g^n\beta$ must be strictly disjoint from β. From the earlier discussion, we see that α must be quasigeodesic, and so $\langle g \rangle$ is quasigeodesic.

Lemma 13.13 : *Every infinite cyclic subgroup of Γ is quasiisometrically embedded.*

Proof : Suppose $g \in \Gamma$ has infinite order, and let α be an axis of g. By Lemma 13.11, there is a periodic path, β, which crosses α. Let $\mathcal{A}$ be the set of Γ-images of α and β. By Lemma 13.12 and the subsequent discussion, we see that $\langle g \rangle$ is quasigeodesic. $\diamondsuit$

We are now in a position to deal with the "euclidean" case:

Proposition 13.14 : *Suppose that Γ contains an infinite order element with $[\mathrm{Comm}(g), \langle g \rangle] = \blacksquare$ ∞. Then Γ contains a subgroup of finite index isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.*

Proof : Let α be an axis of g. We can find $h \in \mathrm{Comm}(g)$ with $h\alpha$ strictly disjoint from α and with α and $h\alpha$ consistently directed. Since $\langle g \rangle$ is quasiconvex, there is some $n > 0$ such that $G = \langle g^n, h \rangle \cong \mathbf{Z} \oplus \mathbf{Z}$.

Now it's fairly easy to see that G must have finite index in Γ. For example, if $[\Gamma, G] = \infty$, then we can find some $k \in \Gamma$ such that $h^m\alpha$ and $kh^n\alpha$ are strictly disjoint for all $m, n \in \mathbf{Z}$. Now it's easily seen that any three arcs from the set $\{h^m\alpha, kh^n\alpha \mid m, n \in \mathbf{Z}\}$ must be nested (since some pair of them will be parallel). We see that α and $k\alpha$ are separated by an infinite set of disjoint images of $\Lambda(\alpha)$ which is clearly a contradiction. $\diamondsuit$

We can now assume that $[\mathrm{Comm}(g) : \langle g \rangle] < \infty$ for every infinite order element, $g \in \Gamma$.

Let $\mathcal{A}$ be a Γ-invariant set of directed periodic arcs with $\mathcal{A}/\Gamma$ finite. Let E be the set of strict parallel classes in $\mathcal{A}$ (i.e. taking account of directions). Thus, $E = \mathcal{A}/\sim^+$, where $\sim^+$ is the relation defined in Section 10. We see that E admits a natural cyclic order, as defined in Section 10. Since we are assuming that Γ is orientation preserving, it follows that Γ preserves this order. Each $\vec{\alpha} \in \mathcal{A}$ determines an ordered pair, $([-\alpha], [\alpha]) \subseteq E$ (where $[.]$ denotes equivalence class). An undirected element, α, determines an unordered pair

$\pi(\alpha) = \{[-\alpha], [\alpha]\} \in \Phi(E)$. Let $\Pi = \{\pi(\alpha) \mid \alpha \in \mathcal{A}\}$. The stabiliser, $\Gamma(\pi)$, of $\pi = \pi(\alpha)$ is precisely the commensurator of any period of α. Thus, $\Gamma(\pi)$ is two-ended. We see that Π is a full Γ-pattern on E.

By Lemma 13.12, we deduce immediately that:

Lemma 13.15 : *Suppose that $\lambda \in \Pi$ and that $g \in \Gamma(\lambda)$ has infinite order. If $\mu \in \Pi$ with $\lambda \times \mu$, then $g\mu : \mu$.* $\diamondsuit$

Lemma 13.16 : *If $\lambda \in \Pi$, then $\{\mu \in \Pi \mid \mu \times \lambda\}/\Gamma(\lambda)$ is finite.*

Proof : Let $\lambda = \pi(\alpha)$. If $\mu = \pi(\beta)$ crosses λ, then $\alpha \times \beta$, and so $\Lambda(\alpha) \cap \Lambda(\beta) \neq \emptyset$. Since the quotient of $\Lambda(\alpha)$ by any period of α is compact, the result follows. $\diamondsuit$

Lemma 13.17 : *The pattern Π satisfies property (F1).*

Proof : Suppose that $\lambda = \pi(\alpha)$ and $\mu = \pi(\beta)$ are distinct elements of Π with $\lambda : \mu$. Let δ be any path connecting α to β in X. Suppose that $\nu = \pi(\gamma) \in \Pi$ with $\lambda - \nu - \mu$. Now there is a bound on how deeply γ can cross α or β, and so it's not hard to see that γ must enter some bounded neighbourhood of δ. By local finiteness of $\mathcal{A}$, the set of possible γ and hence ν is finite. $\diamondsuit$

Lemma 13.18 : *The pattern Π satisfies (F2).*

Proof : This uses Lemmas 13.15, 13.16 and 13.17, exactly as in [Bo]. The idea is that if there exist $\lambda \neq \mu \in \Pi$ with $\Pi' = \{\nu \in \Pi \mid \nu \times \lambda, \nu \times \mu\}$ infinite, then one uses Lemma 13.15 to interpolate between the different elements of Π'. Thus, using Lemma 13.16 applied to the axes λ and μ, and the fact that the axes corresponding to different elements of Π diverge in X, one can find two fixed elements of Π which are separated by arbitrarily many other elements, contradicting (F1). $\diamondsuit$

Proof of Theorem 13.3 : By Lemma 13.14, we can assume that the commensurator of every infinite order element is two-ended. By Theorem 12.9, X contains a periodic arc, α. By Lemma 13.11, there is a periodic arc, β, which crosses α. Let $\mathcal{A}$ be the set of Γ-images of α and β. Let E be the cyclically ordered set and let $\Pi \subseteq \Phi(E)$ be the full Γ-pattern constructed above. By Lemmas 13.15, 13.17, 13.18 and 13.6 we see that Π satisfies the hypotheses of Lemma 13.9. If Π is cross-connected, we can thus apply Lemma 13.5 and Theorem 13.4 to deduce that Γ is a virtual surface group.

If Π is not cross-connected, we have to work a bit harder. Let P be a cross-connected component of Π, and let $\{g_1, \ldots, g_n\}$ be a finite generating set of Γ. Now, by Lemma 13.7, we can canonically embed the set of cross-connected components to a simplicial tree Σ. Now, Γ acts simplicially on Σ, and it is readily checked that this action is non-trivial. Now, by a standard argument of Bass-Serre theory, we see that if $g_i P \neq P$, then there is an infinite order element, $h_i \in \Gamma$, such that $g_i P$ separates P from $h_i P$. Let γ_i be an axis

of h_i. It is easily seen that γ_i crosses elements of $\mathcal{A}$ corresponding to both P and $g_i P$. Now, let $\mathcal{B}$ consist of $\mathcal{A}$ together with all Γ-images of each γ_i arising in this way. This gives us a larger Γ-pattern on a new cyclically ordered set. This time, we can be sure that the pattern is cross-connected, so Lemma 13.5 applies. $\qquad\qquad\diamondsuit$

14. Characterisations.

In this section we bring together various results from previous sections and describe a number of characterisations of virtual surface groups. In particular, we give proofs of the main results stated in the introduction.

Recall that by a *virtual surface group* we mean a group, Γ, with a finite index subgroup isomorphic to the fundamental group of a closed surface other than the 2-sphere or projective plane.

Here are some other descriptions which turn out to be equivalent:

(S): Γ is a virtual surface group.

(G): Γ acts isometrically, properly discontinuously cocompactly on either the euclidean or the hyperbolic plane.

(C): Γ is either virtually $\mathbf{Z} \oplus \mathbf{Z}$ or else acts as a uniform convergence group on the circle.

(R): Γ is finitely generated and quasiisometric to a complete (riemannian) plane.

(L): Γ is finitely generated and one-ended, contains an element of infinite order, and every infinite order element of Γ has codimension-one.

(I): Γ is finitely presented, one-ended and semistable at infinity with $\pi_1^\infty(\Gamma) \cong \mathbf{Z}$.

(E): Γ is quasiisometric to a locally finite uniformly acyclic 2-complex which is homologically semistable over $\mathbf{Z}$, and whose end homology over $\mathbf{Z}$ is infinite cyclic.

(P): Γ is quasiisometric to a locally finite 2-complex which is planar over $\mathbf{Z}$.

(D$_\mathbf{Q}$): Γ is $PD(2)$ over $\mathbf{Q}$.

(H$_\mathbf{Q}$): Γ is FP_2 over $\mathbf{Q}$ and $H^2(\Gamma, \mathbf{Q}\Gamma)$ contains a 1-dimensional Γ-invariant subspace.

In either of the properties (D) and (H), we can replace $\mathbf{Q}$ by an arbitrary field, $\mathbf{F}$, to get properties which are, in general, stronger than (S). Note, in particular, (H$_\mathbf{F}$) $\Rightarrow$ (S) is precisely Theorem 0.1, from which most of the implications will follow.

We should begin by commenting on some of the above conditions.

Note that a consequence of property (G) is that Γ has a unique maximal finite normal subgroup — namely the kernel of the action of Γ. The quotient of Γ by this subgroup is an orbifold group, i.e. the orbifold fundamental group of a closed (euclidean or hyperbolic) 2-orbifold. Such orbifolds are easily classified, and the topological type is determined by the fundamental group, and hence by Γ.

In (C), the term "uniform convergence group" means that Γ by homeomorphism on S^1 such the induced action on the space of distinct triples (i.e. $S^1 \times S^1 \times S^1 \setminus \{(x, y, z) \mid x \neq y \neq z \neq x\}$) is properly discontinuous and cocompact. (The general theory of convergence groups was developed by Gehring and Martin [GeM].)

In (R), "complete riemannian plane" means any complete riemannian 2-manifold homeomorphic to $\mathbf{R}^2$. In fact, any complete path-metric space homeomorphic to $\mathbf{R}^2$ will do. Such a metric is necessarily proper, i.e. closed bounded sets are compact. We can get away with even less. An analysis of the argument shows that all we really require is a proper metric such that any pair of points lies in some connected subset whose diameter is bounded by some function of the distance between them. One certainly needs some hypothesis of this nature: one can always put a stupid proper metric on the plane so that it is quasiisometric to any given one-ended group.

Condition (L) is the same as the conclusion of Proposition 13.2. We say that an infinite order element, $g \in \Gamma$, has "codimension-one" if the subgroup $\langle g \rangle$ does, i.e. if $X/\langle g^n \rangle$ has more than one end for some n and some (hence any) Cayley graph, X, of Γ. (In fact we can take $n = 2$ in the present situation.)

In (I), the term "semistable at infinity" was defined by Mihalik [Mi]. For a one-ended group, Γ, it means that some (hence any) simply connected 2-complex on which Γ acts properly discontinuously cocompactly has the property that any two rays are properly homotopic. Here, $\pi_1^\infty(\Gamma)$ denotes the fundamental group at infinity. We shall elaborate on these notions shortly.

In properties (E) and (P), we use the term "quaiisometric" to mean that Γ is finitely generated, and that some (hence every) Cayley graph of Γ is quasiisometric to the 1-skeleton of a locally finite 2-complex X with the given properties. We always assume that there is some bound on the lengths of the boundaries of 2-cells of X. All the properties mentioned can be seen to be quasiisometry invariant, or more precisely, the property that a graph is the 1-skeleton of some 2-complex satisfying these conditions is quasiisometry invariant. Moreover they are also invariant under the addition of extra 2-cells. Thus, starting with a Cayley graph of Γ and adding cells to all circuits of length at most some sufficiently large constant, there is no loss in assuming that Γ in fact acts properly discontinuously cocompactly on X itself.

In both (E) and (P) we are assuming that X is uniformly acyclic over $\mathbf{Z}$, as defined in Section 4. To say that X is planar over $\mathbf{Z}$, we mean that it admits an integral winding number satisfying axioms (W1)–(W3). What we call "homological semistability at infinity" is the obvious homological equivalent of semistability, which we describe later. By the *end homology* of X, we mean the inverse limit of the groups $H_1(K^C, \mathbf{Z})$ as K varies over all compact subsets of X.

We can obtain variations of properties (E) and (P), denoted $(E_\mathbf{F})$ and $(P_\mathbf{F})$, by replacing $\mathbf{Z}$ everywhere by an arbitrary field $\mathbf{F}$. In $(E_\mathbf{F})$, the clause about semistability becomes redundant (Lemma 14.3). We have $(E_\mathbf{F}) \Rightarrow (P_\mathbf{F}) \Rightarrow (S)$, and the converses hold if $\mathbf{F} = \mathbf{Q}$.

The fact that $(S) \Rightarrow (G)$ uses the solution to Nielsen realisation problem for finite groups [Ke]. The converse can be seen using Selberg's Lemma, or by a direct geometric argument.

It's more or less clear that (S) implies all the other properties listed. (One can get from (S) to (C) bypassing (G) using the fact that, in the non-euclidean case, Γ is hyperbolic with boundary S^1.) The implication (C) $\Rightarrow$ (S) is the result of the analysis of convergence actions on the circle pioneered by Tukia [T], and completed independently by Gabai [Ga] and Casson and Jungreis [CJ]. (In fact, the argument of [Ga] proves (G) directly, without

passing via (S).) The proofs that the other properties imply (S) all pass via this result.

The fact that (R) $\Rightarrow$ (S) was shown by Mess [Me]. In that paper, the euclidean case is separated from the hyperbolic case by the recurrence or transience of Brownian motion on the riemannian plane. The euclidean case is dealt with using the result of Varopoulos (see [V] and the references therein, or [W]), namely that the random walk on a finitely generated group is recurrent if and only if it virtually abelian of rank at most 2. This in turn relies on Gromov's result [Gr] on groups of polynomial growth, which rests ultimately on the solution to Hilbert's fifth problem [MoZ]. In other words, it calls for some high-powered machinery. An approach to Mess's result which uses [Gr] directly has recently been described by Maillot [Ma].

The fact that (L) $\Rightarrow$ (S) is shown in [Bo] using results and ideas from [DuSa], [DuSw] and [Sw].

The main result of this paper shows that $(H_{\mathbf{F}}) \Rightarrow (P_{\mathbf{F}}) \Rightarrow$ (C). The remaining implications are all fairly elementary as we describe shortly.

We note that an alternative approach to $(P_{\mathbf{Q}}) \Rightarrow$ (C) (at least in the finitely presented case) has been proposed by Kleiner. This approach also makes use of [Gr], though given the existence of an infinite order element (Theorem 12.9 of this paper) this can be avoided. It has also been observed by Dunwoody and Swenson [DuSw] that their result also proves $(D_{\mathbf{Q}}) \Rightarrow$ (C), in the finitely presented case, again under the assumption of the existence of an element of infinite order.

In the rest of this section, we set about explaining the proofs of the remaining implications. We begin with:

Proof of Theorem 0.1 : Suppose Γ satisfies $(H_{\mathbf{F}})$. By Proposition 2.2, Γ acts properly discontinuously on a a metric 2-complex which is planar over $\mathbf{F}$. In other words Γ admits a winding number with values in $\mathbf{F}$ satisfying (W1)–(W3). By the results of Section 7, we can lift to a rational integral winding number. Thus, by Theorem 13.3, Γ is a virtual surface group. $\diamondsuit$

Note that we have passed via $(P_{\mathbf{F}})$. In other words, we have shown also that $(P_{\mathbf{F}}) \Rightarrow$ (S).

Next, we move on to property (I). Let X be any locally finite 2-complex with a bound on the lengths of boundaries of 2-cells. We assume that X is uniformly simply connected, i.e., every closed curve in X bounds a disc whose diameter is bounded as a function of the diameter (or equivalently, length) of the curve. We shall restrict attention to the case where X is one-ended. Thus X is semistable at infinity if and only if every two rays are properly homotopic. This is equivalent to the statement that for all compact $K \subseteq X$, there is a compact $L \supseteq K$ such that for all compact $M \supseteq L$, the images of $\pi_1(M^C)$ and $\pi_1(L^C)$ in $\pi_1(K^C)$ are equal (see [Mi]). This allows us to define the fundamental group at infinity, $\pi_1^\infty(X)$ as the inverse limit of the system $(\pi_1(K^C))_K$ as K varies over all compact subsets of X. The proper homotopy equivalence of rays in X tells us that we don't have to worry about basepoints. These definitions are all quasiisometry invariant (for the 1-skeleton in the sense described earlier). We therefore get a definition of semistability for any finitely presented group, Γ, and of $\pi_1^\infty(\Gamma)$. (It is an open problem as to whether every finitely presented group is semistable at infinity.)

Let $(K_i)_{i \in \mathbf{N}}$ be a compact exhaustion of X. Write $G_i = \pi_1(K_i^C)$ and $G_\infty = \pi_1^\infty(X)$. Thus, G_∞ is the inverse limit of the system $(G_i)_i$. By semistability, we can assume that for all i, the images of G_j in G_i are all equal for $j > i$. Putting H_i equal to this image, we can express G_∞ as the inverse limit of the system $(H_i)_i$, where all the connecting maps are surjective. If $G_\infty \cong \mathbf{Z}$, we see that $H_i \cong \mathbf{Z}$ for all sufficiently large i. (Otherwise each H_i would be finite, giving the contradiction that $\mathbf{Z}$ is profinite.) We conclude:

Lemma 14.1 : *If X is semistable at infinity and $\pi_1^\infty(X) \cong \mathbf{Z}$, then there exist compact sets, $K \subseteq L \subseteq X$ such that for all $M \supseteq L$, the image of $\pi_1(M^C)$ in $\pi_1(K^C)$ is infinite cyclic.* $\diamond$

One can also define homological versions of semistability. Suppose $\mathbf{A}$ is a ring and that X is a locally finite one-ended uniformly acyclic 2-complex (with a bound on the lengths of boundaries of 2-cells). We say that X is *homologically semistable (at infinity) over* $\mathbf{A}$ if for all compact $K \subseteq X$, there is some compact $L \supseteq K$ such that for all compact $M \supseteq L$, the image of $H_1(M^C; \mathbf{A})$ in $H_1(K^C; \mathbf{A})$ equals the image of $H_1(L^C; \mathbf{A})$ in $H_1(K^C; \mathbf{A})$. We define the *end homology* as the inverse limit of the system $(H_1(K^C; \mathbf{A}))_K$ as K ranges over all compact subsets of X. We denote it by $H_1^\infty(X; \mathbf{A})$. (Note that the definition of end homology makes sense even if the semistability condition is dropped — for homology, we don't have to worry about basepoints.) Also these properties are again quasiisometry invariant, and so make sense for any group which acts property discontinuously cocompactly on an $\mathbf{A}$-acyclic 2-complex (which is the same as FP_2 over $\mathbf{A}$ if $\mathbf{A}$ is $\mathbf{Z}$ or any field).

From Lemma 14.1, we deduce immediately:

Lemma 14.2 : *If X is uniformly simply connected and semistable at infinity with $\pi_1^\infty(X) \cong \mathbf{Z}$, then it's also homologically semistable over $\mathbf{Z}$, and $H_1^\infty(X; \mathbf{Z}) \cong \mathbf{Z}$.* $\diamond$

Lemma 14.3 : *Suppose $\mathbf{F}$ is any field, and X is uniformly acyclic. Then X is homologically semistable over $\mathbf{F}$.*

Proof : Suppose $K \subseteq X$ is compact. Applying Mayer-Vietoris to $X = K \cup K^C$, we see that $H_1(K^C; \mathbf{F})$ is finite-dimensional. Choose $L \supseteq K$ so as to minimise the dimension of the image of $H_1(L^C; \mathbf{F})$ in $H_1(K^C; \mathbf{F})$. If $M \supseteq K$ is compact, then the image of $H_1(M^C; \mathbf{F})$ in $H_1(K^C; \mathbf{F})$ is equal to that of $H_1(L^C; \mathbf{F})$. $\diamond$

Given compact $K \subseteq X$, let V_K denote the image of $H_1^\infty(X; \mathbf{F})$ in $H_1(K^C; \mathbf{F})$ under that natural map. Suppose $H_1^\infty(X; \mathbf{F})$ is finite dimensional. Choose K so as to maximise the dimension of V_K. We see that the natural map of $H_1^\infty(X; \mathbf{F})$ into $H_1(K^C; \mathbf{F})$ is injective, so that V_K can be naturally identified with $H_1^\infty(X; \mathbf{F})$. The same goes for any compact set containing K. Putting this together with semistability (Lemma 14.2), we conclude that:

Lemma 14.4 : *Suppose that X is $\mathbf{F}$-acyclic and $H_1^\infty(X; \mathbf{F})$ is finite-dimensional. Then $(\exists K_0)(\forall K_1 \supseteq K_0)(\exists L_0 \supseteq K_1)(\forall L_1 \supseteq L_0)$ with $K_0, K_1, L_0, L_1 \subseteq X$ compact, the image of $H_1(L_1; \mathbf{F})$ in $H_1(K_1; \mathbf{F})$ is naturally identified with $H_1^\infty(X; \mathbf{F})$.* $\diamond$

Now, if $H_1(X; \mathbf{Z}) = 0$, then by the universal coefficient theorem, $H_1(X; \mathbf{F}) = 0$ for any field, $\mathbf{F}$. Suppose X is homologically semistable over $\mathbf{Z}$. If $H_1^\infty(X; \mathbf{Z}) \cong \mathbf{Z}$, then since $\mathbf{Z}$ is not profinite, by the same argument as for fundamental groups, we see that the same conclusions as Lemma 14.4 holds with $\mathbf{Z}$ replacing $\mathbf{F}$. Now, again using the universal coefficient theorem, we deduce:

Lemma 14.5 : *If X is $\mathbf{Z}$-acyclic and homologically semistable over $\mathbf{Z}$, and $H_1^\infty(X; \mathbf{Z}) \cong \mathbf{Z}$, then $H_1^\infty(X; \mathbf{F}) \cong \mathbf{F}$ for any field $\mathbf{F}$.* $\diamond$

We shall now gather these facts together to deduce (R) $\Rightarrow$ (I) $\Rightarrow$ (E) $\Rightarrow$ (E$_\mathbf{F}$) $\Rightarrow$ (P$_\mathbf{F}$) $\Rightarrow$ (C) $\Rightarrow$ (S) for any field $\mathbf{F}$. Of course it will be clear that one can pass more directly from any of (R), (I) or (E) to a planar complex with integral winding numbers.

To prove (R) $\Rightarrow$ (I), we first make some preliminary observations. Recall that a path-metric space is *taut* if every point lies on (or equivalently, is a bounded distance from) a uniform biinfinite path with uniform parameters. (Here we need only consider quasigeodesics.) We say that a space is *uniformly simply connected* if it satisfies some isodiametric inequality; in other words, every loop, γ, bounds (the continuous image of) a disc whose diameter is bounded as a function of the diameter of γ. Clearly, the property of being taut is a quasiisometry invariant of path-metric spaces, whereas that of being uniformly simply connected is not.

Every simply connected complex which admits a cocompact group action is both taut and uniformly simply connected. Also, any taut path-metric on the plane is uniformly simply connected. To see that latter statement, first note that it is enough to consider simple closed curves γ. Now any such curve, γ, bounds an embedded disc, D. If $x \in D$, then x lies on a biinfinite uniform path, α, with fixed parameters. Each ray of α emanating from x must intersect γ. The distance between these intersections is bounded by diam(D), and hence places an upper bound on the distance between x and γ.

Now suppose that Γ is a finitely generated group which is quasiisometric to a plane, R, with a complete path-metric. It is a simple exercise to triangulate R such that diameters of the 2-simplexes are bounded — start with any topological triangulation and take a sufficiently fine subdivision. (We do not assume that the edges of the triangulation are rectifiable, or that there is any lower bound on the distance between distinct vertices. However, this can be achieved, at least in the riemannian case [Ma].)

Let K be a Cayley graph of Γ, and let $\phi : K \longrightarrow R$ and $\psi : R \longrightarrow K$ be quasiinverse quasiisometries. We can assume that ϕ and ψ both map vertices to vertices.

Suppose that β is a loop in K. Now $\phi\beta$ is a bounded distance from a loop, γ, in the 1-skeleton of R. This bounds a simplicial disc, D, in R. We map the vertices of this disc back to K using ψ, and then extend over the 1-skeleton by mapping edges of D to geodesics in R. The image of ∂D will be a bounded distance from the original loop β. After connecting vertices of this image with nearby vertices of β, we end up spanning β by the continuous image of the 1-skeleton of a triangulation of the disc, in such a way that the length of the boundary of each 2-simplex is bounded, by some constant, k, say. We deduce that if we attach a 2-cell to each circuit of K of length at most k, then we construct a (locally finite) 2-complex, Σ, which is simply connected. Now Γ acts cocompactly on Σ. We have thus shown that Γ is finitely presented. From this point on, we can assume that

Σ is a simplicial complex, and that K is the 1-skeleton of Σ.

We now know that Σ is taut and uniformly simply connected. Thus R is taut, and since it is a plane, it is also uniformly simply connected.

Now we can realise the quasiisometry $\psi : R \longrightarrow \Sigma$ as a proper simplicial map, possibly at the cost of subdividing R. We can do this by first mapping in the vertices, then mapping each edge of R to a geodesic in the 1-skeleton of Σ, and then extending to each 2-simplex of R using the fact that Σ is uniformly simply connected to bound the diameter of their images. This modified map is a bounded distance from the original, and hence also a quasiisometry. At this point, we need the fact that R is complete to see that ψ is proper.

Now we know that R is semistable at infinity and that $\pi_1^\infty(R) \cong \mathbf{Z}$ (since these are just topological notions). It is now a simple exercise to check that these properties are also true of Σ, using the fact that R is uniformly simply connected to push homotopies in Σ a bounded distance into R. We have shown that $\pi_1^\infty(\Gamma) \cong \pi_1^\infty(X) \cong \mathbf{Z}$ as required. This finally shows that (R) $\Rightarrow$ (I).

Now, (I) $\Rightarrow$ (E) and (E) $\Rightarrow$ ($E_{\mathbf{F}}$) follow respectively from Lemmas 14.2 and 14.5.

To deduce ($E_{\mathbf{F}}$) $\Rightarrow$ ($P_{\mathbf{F}}$), let X be an $\mathbf{F}$-acyclic 2-complex on which Γ acts properly discontinuously cocompactly. Thus, $H_1^\infty(X; \mathbf{F}) \cong \mathbf{F}$. For simplicity, let's assume that there is only one Γ-orbit of vertex. Given any vertex, $x \in X$, applying Lemma 14.4, we find that there exist $r \geq s \geq 0$ such that the image of $H_1(N(A,r)^C; \mathbf{F})$ in $H_1(N(A,s)^C; \mathbf{F})$ can be canonically identified with $H_1^\infty(X; \mathbf{F})$, where A can be x, or any edge incident on x. By Γ-equivariance, we see that the same is true for any vertex of X.

Now choose any non-zero element $\zeta \in H_1^\infty(X; \mathbf{F})$. Given a vertex $x \in X$ then a loop, β, with $\rho(x, \rho) \geq r$ determines an element of $H_1(N(x,r)^C; \mathbf{F})$ whose image in $H_1(N(x,s)^C; \mathbf{F})$ gives us an element, $\omega\zeta$, of $H_1^\infty(X; \mathbf{F})$. We define $\omega(x, \beta) = \omega$.

Properties (W1) and (W3) are immediate. To see (W2), suppose that $x, y \in X$ are adjacent vertices, connected by an edge, e. Suppose β is a loop with $\rho(x, \beta) \geq r$ and $\rho(y, \beta) \geq r$. Then, β determines an element of $H_1(N(e,r)^C; \mathbf{F})$. The images of this element in $H_1(N(e,s)^C; \mathbf{F})$, $H_1(N(x,s)^C; \mathbf{F})$ and $H_1(N(y,s)^C; \mathbf{F})$ all correspond to the same element of $H_1^\infty(X; \mathbf{F})$. Thus, by definition, $\omega(x, \beta) = \omega(y, \beta)$.

We see that ω is a winding number with separation constant, r. Thus X is planar, proving property ($P_{\mathbf{F}}$).

This discussion leaves open the question as to whether the implication (E) $\Rightarrow$ (S) remains true if we drop the assumption of homological semistability from (E).

15. The Seifert Conjecture.

The Seifert conjecture, proved in [T,Me,Ga,CJ], states that if M is a closed (orientable) irreducible 3-manifold whose fundamental group contains an infinite cyclic normal subgroup, then M is a Seifert fibred space. The results of Section 14 enable us to give a homological version of this result. Applying the result of [Sco1], one can recover the original Seifert conjecture.

Specifically, we shall show:

Theorem 15.1 : *Let* **F** *be a field, and suppose that* Γ *is a group which is* FP_3 *over* **F***, and such that* $H^3(\Gamma; \mathbf{F}\Gamma)$ *contains a 1-dimensional invariant subspace. Suppose that* Γ *contains an infinite cyclic normal subgroup. Then the quotient of* Γ *by this subgroup is a virtual surface group.*

The proof Theorem 15.1 uses the LHS spectral sequence (see for example [Br]). A very similar argument is used in [H2]. I am grateful to Ian Leary for explaining to me how this works.

Proof of Theorem 15.1 : Suppose N is a normal subgroup of a group, Γ, with quotient $G = \Gamma/N$. If M is any left $\mathbf{F}\Gamma$-module, the LHS spectral sequence for cohomology has second page $E_2^{ij} = H^i(G; H^j(N; M))$ and converges to $H^{i+j}(\Gamma; M)$. Moreover, if M is a bimodule, so that $H^i(\Gamma, M)$ is a right $\mathbf{F}\Gamma$-module, then we get a spectral sequence of right $\mathbf{F}\Gamma$-modules. In our case, $N \cong \mathbf{Z}$ and $M = \mathbf{F}\Gamma$. Thus $H^j(N; M) = H^j(\mathbf{Z}; \mathbf{F}\Gamma)$ which, as a right $\mathbf{F}\Gamma$-module, is easily seen to be equal to $\mathbf{F}G$ if $j = 1$ and 0 if $j \neq 1$. Thus, the spectral sequence stabilises immediately, with $H^i(G; \mathbf{F}G)$ in row $j = 1$, and 0 everywhere else. Thus, for all $j \geq 0$, we obtain $H^i(G; \mathbf{F}G) \cong H^{i+1}(\Gamma; \mathbf{F}\Gamma)$. This is an isomorphism of right $\mathbf{F}\Gamma$-modules. In particular, we deduce that $H^2(G; \mathbf{F}G)$ has a 1-dimensional G-invariant subspace.

Now it is shown in [Bi] (making similar use of the LHS spectral sequence for homology, and applying the Bieri-Eckmann finiteness criterion) that G must be FP_2 (in fact, FP_3) over $\mathbf{F}$ (see [Bi, Proposition 2.7]). Thus, the hypotheses of Theorem 0.1 are satisfied, and we deduce that G is a virtual surface group. $\diamond$

As a result, we may deduce:

Corollary 15.2 : *If* Γ *is a torsion-free group satisfying the hypotheses of Theorem 15.1, then it is the fundamental group of a Seifert fibred 3-manifold.* $\diamond$

In the non-orientable case, we are using the general definition of a Seifert fibred 3-manifold, as found in [Sco2] for example. In other words, we are allowing for the possibility of a neighbourhood of a fibre being a solid Klein bottle, so that the base orbifold may have circular mirrors.

To deduce Corollary 15.2 from Theorem 15.1, we need the following:

Lemma 15.3 : *Suppose* Γ *is a torsion-free group with infinite cyclic normal subgroup,* $N \lhd \Gamma$*, such that* Γ/N *is planar. Then* Γ *is the fundamental group of a Seifert fibred 3-manifold.*

This result seems to be folklore, and can be proven by explicit construction (if we start from the characterisation (G) of planar groups in Section 14). In the orientable case, it is stated in [Sco1]. The case where N is central, and the quotient is fuchsian is given in [Z] (see Theorem 63.1). However, since I know of no explicit reference for the general case, I outline an argument below. We shall make much use of the fact that a torsion free virtually cyclic group is infinite cyclic.

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Proof of Lemma 15.3 : First, we note that we can assume that the quotient group is an orbifold group. To see this, let $\phi : \Gamma \longrightarrow G$ be the quotient map. Let K be the maximal finite normal subgroup of G, and let $\psi : G \longrightarrow G/K = H$ be the quotient map. Now, H is an orbifold group, and $\ker\phi$ had finite index in $\phi^{-1}K = \ker(\psi \circ \phi)$. Now, $\phi^{-1}K$ is torsion-free and two-ended, and thus infinite cyclic. Thus, replacing $\phi : \Gamma \longrightarrow G$ by $\psi \circ \phi \longrightarrow H$, we get an orbifold group quotient as claimed. Note that any finite subgroup of the quotient must be cyclic (since its preimage in Γ is infinite cyclic).

Now, let $\phi : \Gamma \longrightarrow G = \Gamma/N$ be the quotient map, and let $G = \pi_1(Q)$ be the orbifold fundamental group of a closed orbifold Q. Now since G has no dihedral subgroups, the singularities of Q consist of (at most) a finite set of cone points, and finite set of circular mirrors. Let $Z(N)$ be the centraliser of N in Γ, and let $G_0 = \phi(Z(N)) \leq G$. Thus, G_0 has index at most 2 in G. Note that a loop around any cone-point lies in G_0.

Let $\alpha_1 \ldots \alpha_m$ be the mirrors of Q, and let $P \subseteq Q \setminus \bigcup_{i=1}^{m} \alpha_i$ be a finite non-empty set containing all the cone points of Q. Let $Q_0 = Q \setminus (P \cup \bigcup_{i=1}^{m} \alpha_i)$, and let $x \in Q_0$ be a fixed basepoint. For each i, let β_i an arc connecting x to α_i, and let $\gamma_1, \ldots, \gamma_n$ be a set of embedded loops based at x, such that the interiors of all the β_i and γ_j are disjoint, and such that each component of $Q \setminus (\bigcup_{i=1}^{m} \beta_i \cup \bigcup_{j=1}^{n} \gamma_j)$ is a disc containing precisely one point of P.

We now construct a 2-complex, D, as follows. Let λ be a circle. For each $i \in \{1, \ldots m\}$, let A_i be an torus or Klein bottle depending on whether or not α_i lies in G_0. Let μ_i be a 2-sided simple closed curve on A_i, and let B_i be a Möbius band, whose core curve is identified with μ_i, and whose boundary is identified with λ. For each $j \in \{1, \ldots, n\}$, let C_j be a torus or Klein bottle depending on whether or not γ_i lies in G_0. We identify a 2-sided simple closed curve on C_j with λ. The union of the A_i, B_i and C_j now gives us our 2-complex, D. Note that D had a natural projection, p, to $\bigcup_{i=1}^{m} \alpha_i \cup \bigcup_{i=1}^{m} \beta_i \cup \bigcup_{j=1}^{n} \gamma_j$.

We now thicken up D to give us a 3-manifold, V, and extend p to V. We can assume that the complement of pV in Q is a disjoint union of open discs each containing one point of P, and that $p|p^{-1}Q_0$ is a fibration with circular fibres. Now, if δ is any simple closed curve in Q_0, then $p^{-1}\delta$ is a torus or Klein bottle depending on whether or not δ lies in G_0. In particular, we see that each boundary component of V is a torus. We can define a surjective homomorphism, $\theta : \pi_1(V) \longrightarrow \Gamma$, with $\ker\theta = \pi_1(\lambda) \leq \pi_1(V)$, and such that $\theta \circ \phi$ is the homomorphism from Γ to G induced by p.

Suppose T is a boundary component of V. Now, T is incompressible, so $\pi_1(T)$ is a subgroup of $\pi_1(V)$. Moreover, $\theta(\pi_1(T))$ is infinite cyclic (being the preimage in Γ of a finite cyclic subgroup of G). We can thus glue a solid torus to T so as to kill $\ker(\theta|\pi_1(T))$. Performing this construction for each boundary component of V, we get a closed 3-manifold, M, with a projection map $p : M \longrightarrow Q$, so that the preimage of every point is a circle, and the preimage of each cone point is a singular fibre. In other words, M is a Seifert fibred space. Now, θ descends to a homomorphism from $\pi_1(M)$ to Γ, which is easily verified to be an isomorphism. $\diamondsuit$

Now, if Γ is $PD(3)$ over $\mathbf{Z}$, then it's necessarily torsion-free. It is also FP_3 over $\mathbf{Z}$ and hence also over $\mathbf{Q}$. Moreover, $H^3(\Gamma; \mathbf{Z}\Gamma) \cong \mathbf{Z}$ and so $H^3(\Gamma; \mathbf{Q}\Gamma) \cong \mathbf{Q}$. Thus Corollary 15.2 applies. We have therefore proven Corollary 0.5.

In the case where the abelianisation of Γ is infinite, Corollary 0.5 was proven by

Hillman [H1]. The main obstacle to generalising this was the issue of whether a torsion group could be $PD(2)$ over $\mathbf{Q}$ — a matter resolved in the negative in this paper. (See also [Kl].)

As a result of Corollary 0.5, we get another proof of the Seifert conjecture.

Corollary 15.4 : *If M is a closed orientable irreducible 3-manifold and $\pi_1(M)$ contains an infinite cyclic normal subgroup, then M is a Seifert fibre space.*

Proof : Since $\pi_2(M) = 0$, $\pi_1(M)$ is $PD(3)$ over $\mathbf{Z}$. By Corollary 0.5, M is homotopy equivalent to a Seifert fibre space, and hence homeomorphic to one by [Sco1]. $\diamondsuit$

Scott's theorem [Sco1] is only stated in the orientable case, so we have reproduced this hypothesis above. However, the difficult (non Haken) case dealt with in [Sco1] is necessarily orientable, so it would seem that this result will generalise without difficultly to the non-orientable case.

In the original proof, Mess [Me] showed that the quotient space was quasiisometric to a complete riemannian plane, and reduced the problem to classifying convergence actions on the plane. This had been partially achieved in [T], and was subsequently completed in [Ga,CJ].

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