A variation of McShane's identity for once-punctured torus bundles.

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1. Introduction.

In [M], Greg McShane described an identity concerning the lengths of simple closed geodesics on a once-punctured torus carrying a complete finite-area hyperbolic structure. The identity states that $\sum(1 + e^{l(\gamma)})^{-1} = 1/2$, where γ ranges over all homotopy classes of simple closed curves, and $l(\gamma)$ is the hyperbolic length of the unique closed geodesic in the homotopy class. The result is true, independently of the hyperbolic structure on the punctured torus. An alternative proof of the identity is given in [B1].

In this paper, we describe a variation of McShane's identity which applies to hyperbolic once-punctured torus bundles.

Suppose M is an orientable complete finite-volume hyperbolic 3-manifold which fibres over the circle with fibre a once-punctured torus. Let $\mathcal{S}$ be the set of closed geodesics in M which correspond to simple closed curves in the fibre. To each $\sigma \in \mathcal{S}$, we may associate its *complex length*, $l(\sigma) \in \mathbf{C}/2\pi i\mathbf{Z}$. Thus $\Re l(\sigma)$ is the (real) hyperbolic length of σ, and $\Im l(\sigma)$ is the rotational component, i.e. the angle through which a normal vector turns when parallelly transported once around the curve. Given an orientation on M, this is well-defined up to a multiple of $2\pi i$. (In fact, in the case of surface bundles, it can be unambiguously defined in $\mathbf{C}$, though we shall not need to worry about that here.) Note, in particular, that $e^{l(\sigma)}$ is well-defined.

To begin with, we claim

Theorem A :

$$\sum_{\sigma \in \mathcal{S}} \frac{1}{1 + e^{l(\sigma)}} = 0.$$

Moreover, the above sum converges absolutely.

Now the curves in $\mathcal{S}$ fall naturally into two classes. One way to explain this is as follows.

Let $\mathcal{C}$ be the set of homotopy classes of non-peripheral simple closed curves on the punctured torus, $\mathbf{T}$. Now, $\mathcal{C}$ can be thought of as the set of rational points in projective lamination space, $\mathcal{P}$, which in this case is a circle [CaB].

The mapping class group of $\mathbf{T}$ acts on $\mathcal{P}$ preserving the set $\mathcal{C}$. The monodromy of M generates an infinite cyclic subgroup of the mapping class group. This subgroup has two fixed points in $\mathcal{P}$, namely the stable and unstable laminations, μ_s and μ_u, of the monodromy. These two points separate $\mathcal{P}$ into two open intervals. Since μ_s and μ_u are irrational points, this gives a natural partition of $\mathcal{C}$ into two subsets, $\mathcal{C}_L$ and $\mathcal{C}_R$, which in turn partitions $\mathcal{S}$ into two subsets $\mathcal{S}_L$ and $\mathcal{S}_R$. If we restrict the sum appearing in Theorem A to one or other of $\mathcal{S}_L$ or $\mathcal{S}_R$, we will get the same answer up to changing sign. This number turns out essentially to be the modulus of the cusp of M.

To explain what we mean, note that M has a single parabolic cusp. In particular, M is (homeomorphic to) the interior of a compact manifold, $M \cup \partial M$, with one toroidal boundary component, ∂M. Now, ∂M carries a natural euclidean structure, well-defined up to similarity, which arises from identifying ∂M with a horocycle. To define the modulus of the cusp, we need to identify a meridian and a longitude for ∂M. A meridian can be defined as the boundary of a fibre in $M \cup \partial M$. Its orientation is determined by the orientation on the fibre.

In order to describe the "longitude", we must first distinguish between what we shall call "positive" and "negative" monodromy. Note that the mapping class group of $\mathbf{T}$ may be identified with the group $SL(2, \mathbf{Z})$. This admits a fixed point free involution sending a matrix A to $-A$. The quotient by this "hyperelliptic" involution is $PSL(2, \mathbf{Z})$. Thus, once-punctured torus bundles occur naturally in pairs, whose monodromies project to the same element of $PSL(2, \mathbf{Z})$. The two manifolds of such a pair are referred to as *sisters*. One sister will have *positive monodromy*, i.e. both its eigenvalues are positive, whereas the other will have *negative monodromy*, i.e. both its eigenvalues are negative. These eigenvalues give rise to the stable and unstable foliations on the unpunctured torus, invariant under this monodromy. In the positive case, the orientations on the fibres are preserved, whereas in the negative case they are reversed. Note that sister manifolds are commensurable in that they have a common double cover.

We shall want the "modulus of the cusp" of sister hyperbolic manifolds to be equal, so it will suffice to define a longitude in case where M has positive monodromy. One way to describe this is to imagine M as the closed torus bundle, M', (with the same monodromy) from which we have removed some circle C, transverse to the fibres. Defining a longitude on ∂M is equivalent to defining a framing of C. Note that the stable and unstable foliations of the monodromy give rise to a transverse pair of codimension-1 foliations of M'. The curve C can be taken to be a component of the intersection of a pair of leaves, one from each of these foliations. Since the monodromy is positive, these leaves define a framing on C. The orientation in the longitude is determined by the orientation on the base circle. (In the case of negative monodromy, we have to go twice around C to get our "framing" to close up. This gives us a curve wrapping twice around ∂M with respect to the meridian. In this case, it might be natural to imagine the longitude as an element of real homology given by a half of this curve.)

If M has positive monodromy, we may represent ∂M as the quotient of $\mathbf{C}$ (with the euclidean metric) by the lattice $\mathbf{Z} \oplus \lambda \mathbf{Z}$, generated by the translations $[\zeta \mapsto \zeta + 1]$ and $[\zeta \mapsto \zeta + \lambda]$ corresponding to the meridian and longitude respectively. We call $\lambda = \lambda(\partial M)$ the *modulus* of the cusp. We can suppose that $\Im \lambda(\partial M) > 0$. If M has negative monodromy, we define $\lambda(\partial M)$ as the modulus of the cusp of its sister.

Theorem B :

$$\sum_{\sigma \in \mathcal{S}_L} \frac{1}{1 + e^{l(\sigma)}} = \pm \lambda(\partial M).$$

The sign depends on our conventions of orientation. Given an orientation on the base circle, we can decide which is the stable and which is the unstable lamination. Given an orientation on the fibre, we get an orientation on $\mathcal{P}$, so it makes sense to say, for example, that the curves in $\mathcal{C}_L$ lie to the right of μ_u and to the left of μ_s. Together, these orientations define an orientation on M and hence on ∂M. Now the orientation on M must also be consistent with the orientation on hyperbolic 3-space, and hence determines the way in which a horocycle should be regarded as a quotient of $\mathbf{C}$. Our precise conventions in these matters determine the sign in Theorem B.

The proofs of Theorems A and B will be along similar lines to those applied to McShane's identity in [B1], and elaborated on in [B2]. We shall omit details from some arguments which would reproduce those given in these papers.

2. A reformulation of the theorems.

Let $\mathbf{T}$ denote the (topological) once-punctured torus, and let $\mathcal{C}$ be the set of homotopy classes of non-peripheral simple closed curves on $\mathbf{T}$. The Teichmüller space of $\mathbf{T}$ may be identified with the hyperbolic plane, $\mathbf{H}^2$, and projective lamination space, $\mathcal{P}$, with the ideal circle of $\mathbf{H}^2$. Consider the regular tessellation of $\mathbf{H}^2$ by ideal triangles, whose ideal vertices form the set of rational points of $\mathcal{P}$. Dual to this triangulation, we have a "binary tree" Σ properly embedded in $\mathbf{H}^2$. Let Ω be the set of complementary regions of Σ, i.e. closures of connected components of $\mathbf{H}^2 \setminus \Sigma$. There is a natural bijection between Ω and the set of rational points of $\mathcal{P}$, which we may identify with $\mathcal{C}$. This bijection has a number of alternative descriptions [B1,B2].

We define a *directed edge*, $\vec{e}$, of Σ as an ordered pair of adjacent vertices of Σ, respectively the *head* and *tail* of $\vec{e}$. We shall (always) write e for the underlying undirected edge. Associated to each $\vec{e}$ are four regions, $X, Y, Z, W \in \Omega$ such that $e = X \cap Y$, and such that $e \cap Z$ and $e \cap W$ are the head and tail of $\vec{e}$, respectively. We shall speak of Z as the region at the head of $\vec{e}$. Moreover, given an orientation on $\mathbf{H}^2$, it makes sense to say that X lies "on the left" of $\vec{e}$, and that Y lies "on the right", where we imagine $\vec{e}$ as pointing upwards.

Now $\Gamma = \pi_1(\mathbf{T})$ is a free group on two generators. If $\vec{e}, X, Y, Z, W$ are as above, then we can choose a free basis, a, b for Γ so that X, Y, Z, W are represented respectively by the elements a, b, ab, ab^{-1}. There is a sign convention involved here. For Z to be represented by ab (rather that ab^{-1}), we choose a, b so that the algebraic intersection number of the corresponding ordered pair of simple closed curves on $\mathbf{T}$ is equal to $+1$. Note that the commutator, $[a, b] = aba^{-1}b^{-1}$, is peripheral.

The mapping class group of $\mathbf{T}$ may be identified as $SL(2, \mathbf{Z})$ which acts naturally on $\mathbf{H}^2$, Σ, Ω, $\mathcal{P}$ and $\mathcal{C}$. In each case, the kernel is given by the hyperelliptic involution, so the induced action of $PSL(2, \mathbf{Z})$ is faithful. An element $H \in SL(2, \mathbf{Z})$ is *hyperbolic* if it has two fixed points μ_u and μ_s in $\mathcal{P}$, namely the stable and unstable laminations. The points are joined by a bi-infinite arc $\beta \subseteq \Sigma$, which is translated by H in the direction of μ_s. The path β can be described combinatorially in terms of the "left-right" decomposition of the matrix H. Note that some conjugate of H in $PSL(2, \mathbf{Z})$ can be written as a product of the matrices $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. This decomposition is well-defined up to cyclic reordering, and the sequence of L's and R's is the same as the periodic sequence of left and right turns of β in Σ. For more details, see for example [BMR].

Note that β partitions Ω into two subsets, Ω_L and Ω_R which lie on the left and right, respectively, of β, where we imagine β as being translated upwards by H. These correspond to the subsets $\mathcal{C}_L$ and $\mathcal{C}_R$ of $\mathcal{C}$ described in Section 1.

Now, if we take a homeomorphism, θ, of $\mathbf{T}$ representing the mapping class H, we may form the mapping torus, M_H, which is given by $(\mathbf{T} \times [0,1])/\!\sim$, where $\sim$ identifies $(x,1)$ with $(\theta(x),0)$ for all $x \in \mathbf{T}$. The manifold M_H has a natural compactification by adjoining a toroidal boundary, ∂M_H. The compactified manifold, $M_H \cup \partial M_H$, has a natural ideal triangulation arising from the left-right decomposition of H. This is described in [FH]. This can be briefly summarised as follows. Associated to each vertex Σ is an ideal triangulation of $\mathbf{T}$ (the edges of which are dual to the simple closed curves corresponding to the three complementary regions incident to the vertex). Moving along an edge in the tree corresponds to performing a (dual) Whitehead move. If we traverse a period of the path β, we get a sequence of Whitehead moves which take us from a given triangulation to its image under θ. Each Whitehead move gives rise to an ideal simplex in $\mathbf{T} \times [0,1]$, and so, after identifying $\mathbf{T} \times \{0\}$ with $\mathbf{T} \times \{1\}$ via the relation $\sim$, we obtain an ideal triangulation of M_H.

This ideal triangulation gives us, in particular, a triangulation of the boundary, ∂M_H. As in [FH], we may describe the combinatorial structure of this triangulation lifted to the universal cover, $\mathbf{R}^2$, of ∂M_H. First, we describe the case of positive monodromy. To do this, consider the bi-infinite sequence of vertices of Σ lying along the arc $\beta \subseteq \Sigma$. This sequence is dual to a sequence of ideal triangles in our tessellation of $\mathbf{H}^2$. The union of these triangles gives a bi-infinite strip invariant under the transformation H. We transfer this strip homeomorphically to the vertical strip $[0,1] \times \mathbf{R}$ in $\mathbf{R}^2$, so that the transformation H is conjugated to the map $[(x,y) \mapsto (x,y+1)]$. We extend this to a triangulation of $\mathbf{R}^2$ by a process of repeated reflection in the pair of vertical lines which form the boundary of this strip (Figure 1). The triangulation of ∂M_H is given by the quotient by the group generated by $[(x,y) \mapsto (x,y+1)]$ and $[(x,y) \mapsto (x+4,y)]$. It's not hard to see that these transformations describe, respectively, the longitude and meridian of ∂M_H. Note that the triangulation is in fact invariant under map $[(x,y) \mapsto (x+2,y)]$. Up to this symmetry, there are two "vertical" lines in the triangulation. The bi-infinite sequence of vertices along one of these lines corresponds to sequence of regions of Ω which meet β and all lie either in Ω_L, or in Ω_R. Two vertices are joined by an edge in this triangulation if and only if the corresponding regions are adjacent. Thus the "vertical" edges correspond to regions meeting on the same side of β, whereas all the other edges correspond to vertices meeting on opposite sides of β. The picture where the monodromy is negative is similar, except that in this case, ∂M_H is given as a quotient of $\mathbf{R}^2$ by the group generated by $[(x,y) \mapsto (x+2,y+1)]$ and $[(x,y) \mapsto (x+4,y)]$. The "longitude" might be thought of as a "half of" the curve given by $[(x,y) \mapsto (x,y+2)]$.

Now, it follows from the work of Thurston [T], that $M = M_H$ admits a complete finite-volume hyperbolic structure. See also [O] for an alternative proof and exposition. This structure is unique by Mostow rigidity. Note that ∂M_H carries a euclidean structure, well defined up to similarity, obtained for example by identifying it with a horocycle in M_H.

In this hyperbolic structure, we may realise each tetrahedron in our ideal triangulation of M as a hyperbolic ideal tetrahedron. In this way we get a "hyperbolic ideal triangulation" of M. This gives rise to a euclidean realisation of the combinatorial triangulation of ∂M_H.

Unfortunately, it is not entirely clear that all the hyperbolic tetrahedra arising in this way have to be positively oriented, so that we get a genuine ideal triangulation in the usual sense. I suspect that this should be the case. In fact, one might conjecture that the triangulation we get coincides with the Delaunay triangulation, i.e. that which arises from the construction of [EP]. However I know of no proof of this written down. For our purposes, this will not formally matter to us. Everything we say will make good sense for negatively oriented simplices, though it is intuitively simpler to imagine our triangulation as embedded.

We should however make one observation which is relevant to the sign convention in Theorem B. Note that there is a natural map from ∂M to itself which sends each combinatorial simplex to the corresponding geometric one. By construction, this map is certainly homotopic to plus or minus the identity, even if we are not sure whether it is, itself, a homeomorphism. Now if we have chosen our conventions of orientation correctly, this map must, in fact, have degree $+1$, i.e. it is indeed homotopic to the identity. This justifies the assertion made after Theorem B, namely that if we insist that $\Im\lambda(\partial M) > 0$, then the sign occurring in Theorem B depends only on our conventions of orientation.

Regarding $\mathbf{T}$ as a fibre of M, we get an identification of $\Gamma = \pi_1(\mathbf{T})$ as a normal subgroup of $\pi_1(M)$. In fact, $\pi_1(M)$ is an HNN extension of Γ with stable letter t so that $tgt^{-1} = H_*(g)$ for all $g \in \Gamma$, where H_* is the automorphism of Γ induced by the monodromy H.

We also get an identification of $\mathcal{S}$ with the quotient, $\Omega/\langle H\rangle$, of Ω under the cyclic group, $\langle H\rangle$, generated by H. Clearly, H respects the partition of Ω as $\Omega_L \sqcup \Omega_R$, and we may identify $\mathcal{S}_L$ with $\Omega_L/\langle H\rangle$ and $\mathcal{S}_R$ with $\Omega_R/\langle H\rangle$.

The hyperbolic structure on M may be described by a representation, $\hat\rho : \pi_1(M) \longrightarrow PSL(2,\mathbf{C})$. It follows from [Cu] that $\hat\rho$ lifts to a representation $\rho : \pi_1(M) \longrightarrow SL(2,\mathbf{C})$. (In fact, this is clear if we allow ourselves to replace M by a finite cyclic cover. This is good enough for proving Theorems A and B, since both the left and right-hand sides of the equations just get multiplied by the order of the cover.) Restricting our attention to the fibre subgroup $\Gamma \triangleleft \pi_1(M)$, we define a map $\phi : \Omega \longrightarrow \mathbf{C}$ by $\phi(X) = \operatorname{tr}\rho(g)$, where $g \in \Gamma$ represents the simple closed curve on $\mathbf{T}$ corresponding to the region $X \in \Omega$. It follows, using trace identities in $SL(2,\mathbf{C})$, that ϕ is a Markoff map, as defined in [B1]. This means that ϕ satisfies "vertex" and "edge relations" for each vertex and edge in Σ. Thus, if a vertex of Σ meets the three complementary regions X, Y and Z, then we have $x^2 + y^2 + z^2 = xyz$, where $x = \phi(X)$, $y = \phi(Y)$ and $z = \phi(Z)$. Also, if e is an edge of E, meeting the four regions X, Y, Z and W, such that $e = X \cap Y$ (so that $e \cap Z$ and $e \cap W$ are the two endpoints of e), then we have $xy = z + w$ where $w = \phi(W)$.

Clearly ϕ is invariant under the $\langle H\rangle$-action, and so gives rise to a well defined map $\Omega/\langle H\rangle \longrightarrow \mathbf{C}$ which we also denote by ϕ. We write $[X]$ for the orbit of X under $\langle H\rangle$.

If $\sigma \in \mathcal{S}$ corresponds to $[X] \in \Omega/\langle H\rangle$, then the complex length, $l(\sigma)$, of σ is determined by the formula $\phi([X]) = 2\cosh(l(\sigma)/2)$. Thus $h(\phi([X])) = 1/(1 + e^{l(\sigma)})$ where $h : \mathbf{C} \setminus [-2,2] \longrightarrow \mathbf{C}$ is defined by $h(\zeta) = \frac{1}{2}(1 - \sqrt{1 - 4/\zeta^2})$. Here we take the square root with positive real part, corresponding to the fact that $\Re l(\sigma) > 0$. Note that $h(\zeta) = O(|\zeta|^{-2})$ as $|\zeta| \to \infty$.

In these terms we may express Theorems A and B, respectively, by the identities

$$\sum_{[X]\in\Omega/\langle H\rangle} h(\phi([X])) = 0$$

and

$$\sum_{[X]\in\Omega_L/\langle H\rangle} h(\phi([X])) = \lambda(\partial M).$$

3. Proofs.

Before we set about proving these statements, let's recall some notation used in [B2]. Suppose that $\vec{e}$ is a directed edge of Σ with underlying edge e. We write $\Omega^0(e) = \{X, Y\} \subseteq \Omega$, where $e = X \cap Y$. If we remove the interior of e from Σ we split Σ into two components. We write Σ^- for the component containing the tail of $\vec{e}$. We write $\Omega^-(\vec{e}) = \{X \in \Omega \mid \partial X \subseteq \Sigma^-\}$, and $\Omega^{0-}(\vec{e}) = \Omega^0(e) \cup \Omega^-(\vec{e})$. We write $-\vec{e}$ for the directed edge obtained by swapping the head and tail of $\vec{e}$.

Recall that β is the $\langle H\rangle$-invariant bi-infinite path in Σ. Thus $\beta/\langle H\rangle$ is a cycle consisting of n edges (where n is the number of components in the left-right decomposition of H). Let $\alpha \subseteq \beta$ be a subarc which is the union of $n-1$ consecutive edges of β. Let C be the set of directed edges, $\vec{e}$, with the property that $e \cap \alpha$ consists of a single vertex of Σ which is the head of $\vec{e}$. We may write C as a disjoint union $C = C_0 \sqcup C_L \sqcup C_R$, where C_0 consists of the two edges of C which lie in β, and C_L (respectively C_R) consists of those edges of C which lie to the left (right) of β. In other words, $C_L = \{\vec{e} \in C \mid \Omega^0(e) \subseteq \Omega_L\}$. (Figure 2.)

Note that Ω_L can be expressed as a union of sets of the form $\Omega^{0-}(g\vec{e})$, as $\vec{e}$ varies in C_L and g varies in $\langle H\rangle$. To be more specific, suppose $X \in \Omega_L$. If $X \cap \beta = \emptyset$, then X occurs in precisely one set of the form $\Omega^-(g\vec{e})$. If $X \cap \beta \neq \emptyset$, then it occurs in two sets of the form $\Omega^0(ge)$. Similarly for Ω_R.

We now return to considering our Markoff map $\phi : \Omega \longrightarrow \mathbf{C}$. Given a directed edge $\vec{e}$ of Σ, we write $\psi(\vec{e}) = \phi(Z)/\phi(X)\phi(Y)$, where $\Omega^0(e) = \{X, Y\}$ and $Z \in \Omega$ is the region at the head of $\vec{e}$. Now the edge relation of ϕ may be expressed as $\psi(\vec{e}) + \psi(-\vec{e}) = 1$. The vertex relation reduces to the statement that if $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are the three directed edges whose heads are at a particular vertex, then $\psi(\vec{e}_1) + \psi(\vec{e}_2) + \psi(\vec{e}_3) = 1$. Applying these relations, we find that $\sum_{\vec{e}\in C} \psi(\vec{e}) = 1$, where C is the finite set of directed edges defined above. This follows as in [B1] since C is a "circular set" of edges. (A set, C, of directed edges is "circular" if there is a finite subtree, T, of Σ such that $\vec{e} \in C$ if and only if e intersects T in the head of $\vec{e}$. In this case, the subtree in question is the arc α.) Now, the two edges $\vec{e}_1, \vec{e}_2 \in C_0$ are images of each other under H except that they are directed in opposite senses, i.e. we can suppose that $H(\vec{e}_1) = -\vec{e}_2$. Since ϕ is $\langle H\rangle$-invariant, it follows that $\psi(\vec{e}_1) + \psi(\vec{e}_2) = 1$. Thus:

Lemma 1 :

$$\sum_{\vec{e}\in C_L\cup C_R} \psi(\vec{e}) = 0.$$

$\diamondsuit$

Lemma 2 : *Suppose that $\vec{e}$ is a directed edge such that $\{X \in \Omega^-(\vec{e}) \mid |\phi(X)| \le 2\}$ is finite. Then $\sum_{X\in\Omega^-(\vec{e})} |\phi(X)|^{-t}$ converges for all $t > 0$.*

Proof : In [B2], it was shown that if $\phi : \Omega \longrightarrow \mathbf{C}$ is a Markoff map, and $\vec{e}$ is a directed edge of Σ such that $\{X \in \Omega^{0-}(\vec{e}) \mid |\phi(X)| \le 2\}$ is finite and $\{X \in \Omega^{0-}(\vec{e}) \mid \phi(X) \in [-2, 2]\}$ is empty, then $\sum_{X\in\Omega^-(\vec{e})} |\phi(X)|^{-t}$ converges for all $t > 0$. Here, the first condition is given as a hypothesis, and the second condition holds since $\rho(\Gamma)$ has no elliptics or accidental parabolics. $\diamondsuit$

Since $h(\zeta) = O(|\zeta|^{-2})$, it follows that $\sum_{X\in\Omega^-(\vec{e})} h(\phi(X))$ converges absolutely. Given the convergence of these sums, the arguments of [B1] can be applied to show the following:

Lemma 3 : *With the hypotheses of Lemma 2, we have*

$$\sum_{X\in\Omega^0(e)} h(\phi(X)) + 2 \sum_{X\in\Omega^-(\vec{e})} h(\phi(X)) = \psi(\vec{e}).$$

Proof : (Sketch) For each $n \in \mathbf{N}$, let C_n be the set of directed edges pointing towards $\vec{e}$, and at a distance n from the tail of $\vec{e}$ and $n+1$ from the head of $\vec{e}$. (In other words, $\vec{f} \in C_n$ if and only if $\vec{e}$ and $\vec{f}$ are at opposite ends of some arc of length $n+2$ in Σ which has endponts at the head of $\vec{e}$ and the tail of $\vec{f}$.) Thus, $C_n \cup \{-\vec{e}\}$ is a circular set, and so $\psi(-\vec{e}) + \sum_{\vec{f}\in C_n} \psi(\vec{f}) = 1$. (See the discussion before Lemma 1.) Since $\psi(\vec{e}) + \psi(-\vec{e}) = 1$, we obtain $\psi(\vec{e}) = \sum_{\vec{f}\in C_n} \psi(\vec{f})$.

Now suppose that n is large, and $\vec{f} \in C_n$. Let $f = X \cap Y$, and $x = \phi(X)$ and $y = \phi(Y)$. We see (from Lemma 2) that either $|x|$ or $|y|$ (or both) will be large. Now, $\psi(\vec{f}) = z/xy$ where $x^2 + y^2 + z^2 = xyz$. Solving for z we obtain $\psi(\vec{f}) = \frac{1}{2}(1 - \sqrt{1 - 4(x^{-2} + y^{-2})})$, which is approximately equal to $h(x) + h(y)$. In other words, $\psi(\vec{f})$ approximately equals $h(\phi(X)) + h(\phi(Y))$.

We see that, up to some error term, if $X \in \Omega^-(\vec{e})$ is at a distance n from $\vec{e}$, then $h(\phi(X))$ contributes twice to the sum equal to $\psi(\vec{e})$, whereas if X is one of the two regions of $\Omega^0(\vec{e})$, then $h(\phi(X))$ contributes once to the sum. We need to verify that the error term tends to 0 as n tends to ∞. This follows exactly as in [B1].

(A more detailed proof of this particular result can be found in [B2].) $\diamondsuit$

We claim that the directed edges of $C_L \cup C_R$ satisfy the hypotheses of Lemma 2:

Lemma 4 : *If $\vec{e} \in C_L \cup C_R$, then $\{X \in \Omega^-(\vec{e}) \mid |\phi(X)| \le 2\}$ is finite.*

Proof : Since M has finite volume, it has a discrete length spectrum. In particular, $\{\sigma \in \mathcal{S} \mid |l(\sigma)| \leq k\}$ is finite for all $k \geq 0$. It follows easily that $\{[X] \in \Omega/\langle H \rangle \mid |\phi(X)| \leq 2\}$ is finite. Now, if $\vec{e} \in C_L \cup C_R$, then $\Omega^-(\vec{e})$ is disjoint from all its images under $\langle H \rangle$, and so the result follows. $\diamondsuit$

Lemma 5 :

$$\sum_{[X] \in \Omega/\langle H \rangle} h(\phi([X])) = 0.$$

Proof : Lemma 4 allows us to take the formula from Lemma 3, and sum over all edges $\vec{e} \in C_L \cup C_R$. Lemma 1 then tells us that the right-hand side of this sum is zero. The discussion preceding Lemma 1 shows that in the sum on the left-hand side, there are exactly two representatives in Ω of each $[X] \in \Omega/\langle H \rangle$. $\diamondsuit$

This proves Theorem A.

In order to prove Theorem B, we need to compute the sum featuring in Lemma 5, restricted to the set $\Omega_L/\langle H \rangle$. We shall assume that the monodromy is positive.

Let's fix an orientation on the meridian of ∂M consistent with the orientation of the fibre. We shall use the upper half-space model of $\mathbf{H}^3$, so that its ideal boundary is identified with the Riemann sphere, $\mathbf{C} \cup \{\infty\}$ which has $PSL(2, \mathbf{C})$ acting in the usual way. We can normalise our representation, $\rho : \pi_1(M) \longrightarrow SL(2, \mathbf{C})$, so that ∞ is a parabolic fixed point. Write $\mathrm{stab}(\infty)$ for the stabliser of ∞ in $\rho(\pi_1(M))$. Thus, the euclidean structure on ∂M is given by $\mathbf{C}/\mathrm{stab}(\infty)$. We can further insist that the meridian in ∂M is given by the transformation $[\zeta \mapsto \zeta + 2]$. This determines the representation, ρ, up to automorphism of $\pi_1(M)$ and conjugacy by a translation of $\mathbf{C}$. From the symmetry of such representations, it turns out that the projection of $\rho(\pi_1(M))$ to $PSL(2, \mathbf{C})$ is normalised by the translation $[\zeta \mapsto \zeta + 1]$ (cf. [J,B2]).

Suppose a, b are free generators of the fibre group Γ such that the ordered pair of simple closed curves on $\mathbf{T}$ that they represent have algebraic intersection number $+1$. Now the commutator, $[a, b] = aba^{-1}b^{-1}$, is peripheral in $\mathbf{T}$, and so represents a meridian of ∂M. After simultaneous conjugacy by an element of Γ, we can suppose that $\rho([a, b])$ describes the translation $[\zeta \mapsto \zeta + 2]$. Since $\mathrm{tr}\, \rho([a, b]) = -2$ it must, in fact, be given by the matrix $\begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}$.

For future reference (see Figure 3), we observe:

Lemma 6 : *$\rho(a)(\infty) - \rho(b)^{-1}(\infty) = z/xy$ where $x = \mathrm{tr}\, \rho(a)$, $y = \mathrm{tr}\, \rho(b)$ and $z = \mathrm{tr}\, \rho(ab)$.*

Proof : The matrices $\rho(a)$ and $\rho(b)$ are determined up to simultaneous conjugacy in $SL(2, \mathbf{C})$ by the traces x, y, z. Given further that $\rho([a, b]) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}$, they are determined up to conjugacy by a translation of $\mathbf{C}$. Thus, for the purposes of computing $\rho(a)(\infty) - \rho(b)^{-1}(\infty)$, we can suppose that they are given by Jørgensen's normalisation

[J]:

$$\rho(a) = \frac{1}{z}\begin{pmatrix} xz - y & x/z \\ xz & y \end{pmatrix} \qquad\qquad \rho(b) = \frac{1}{z}\begin{pmatrix} yz - x & -y/z \\ -yz & x \end{pmatrix},$$

since we may verify that these matrices satisfy the above criteria. Now, $\rho(a)(\infty) = \frac{xz-y}{xz} = 1 - \frac{y}{xz}$ and $\rho(b)^{-1}(\infty) = \frac{x}{yz}$. The result follows by applying the trace identity $x^2+y^2+z^2 = xyz$. $\diamondsuit$

Now, to each region $X \in \Omega$, we shall associate a parabolic fixed point $p(X) \in \mathbf{C}$, which is well-defined up to the action of $[\zeta \mapsto \zeta + 1]$. This is done as follows. We know that X corresponds to some simple closed curve $\gamma(X)$ on $\mathbf{T}$. Let $\delta(X)$ be an arc on $\mathbf{T}$ with both endpoints at the puncture such that $\gamma(X) \cap \delta(X) = \emptyset$. The homotopy class of $\delta(X)$ relative to its endpoints is well defined. We are identifying $\mathbf{T}$ with a fibre of M, so $\mathbf{T}$ is naturally homotopy equivalent to the infinite cyclic cover $\mathbf{H}^3/\rho(\Gamma)$. Under this equivalence, $\delta(X)$, has a unique realisation as a bi-infinite geodesic $\Delta(X)$ in $\mathbf{H}^3/\rho(\Gamma)$. Choose a lift of $\Delta(X)$ to $\mathbf{H}^3$ which has ∞ as an endpoint, and let $p(X)$ be the other endpoint. Any other choice of lift would give us an image of $p(X)$ under the cyclic action generated by $[\zeta \mapsto \zeta + 1]$.

In Section 2, we described the triangulation of the euclidean plane, identified with $\mathbf{C}$, arising as the lift of a triangulation of ∂M. In the combinatorial picture, which we described in $\mathbf{R}^2$, each vertical line corresponds to the bi-infinite sequence, $(X_i)_{i\in\mathbf{Z}}$, of regions either of Ω_L or of Ω_R which meet β. Without loss of generality, let's assume that the X_i all belong to Ω_L. It's not hard to see that in the geometric triangulation of $\mathbf{C}$, the vertex corresponding to X_i is given by one of the images of $p(X_i)$ under the action generated by $[\zeta \mapsto \zeta + 1]$, so we may as well assume that it actually equals $p_i = p(X_i)$. (Note that the choice of p_i naturally determines that of p_{i-1} and p_{i+1}, and so, inductively, p_j for all $j \in \mathbf{Z}$.) Now the sequence (p_i) is periodic under the translation corresponding to the longitude of ∂M. This translation is given by $[\zeta \mapsto \zeta + 2\lambda]$, where $\lambda = \lambda(\partial M)$ is the modulus of the cusp. This corresponds to the action of $\langle H \rangle$ on Ω which has the effect of shifting the sequence (X_i). Let m be the number of steps through which this sequence is shifted. Thus, $2\lambda = p_m - p_0 = \sum_{i=1}^{m}(p_i - p_{i-1})$. We thus want to compute the numbers $p_i - p_{i-1}$.

Let $\vec{e}_i$ be the directed edge given by $X_i \cap X_{i-1}$, whose head lies in β. Now we can assume that the set C_L (as in Lemma 1) consists precisely of the edges $\vec{e}_1, \ldots, \vec{e}_m$.

Fix some $i \in \{1, \ldots, m\}$. Let $X = X_{i-1}$, $Y = X_i$, and let Z be the region at the head of $\vec{e}_i$. As described earlier, we can find free generators a, b for Γ which correspond, respectively, to the regions X and Y. Moreover, we can suppose that a and b satisfy the hypotheses of Lemma 6 (as described immediately before the statement of the lemma). Also, we have that Z is represented by ab. (Note that we need to be careful about sign conventions and orientation. We have arranged that X and Y are on the left and right, respectively, of $\vec{e}_i$, and that the ordered pair a, b has positive algebraic intersection number on $\mathbf{T}$.) Now one can figure out (see Figure 3) that, up to a simultaneous translation of the form $[\zeta \mapsto \zeta + k]$ with $k \in \mathbf{Z}$, the points p_{i-1}, p_i are given by $p_{i-1} = \rho(b)^{-1}(\infty)$ and $p_i = \rho(a)(\infty)$. Thus by Lemma 6, we have $p_i - p_{i-1} = z/xy$, where $x = \operatorname{tr}\rho(a) = \phi(X)$, $y = \operatorname{tr}\rho(b) = \phi(Y)$ and $z = \operatorname{tr}\rho(ab) = \phi(Z)$. Thus $p_i - p_{i-1} = \psi(\vec{e}_i)$.

Using Lemma 3, as in the proof of Lemma 5, it follows that

$$\lambda = \frac{p_m - p_0}{2} = \frac{1}{2}\sum_{i=1}^{m}(p_i - p_{i-1}) = \frac{1}{2}\sum_{\vec{e}\in C_L}\psi(\vec{e}) = \sum_{[X]\in\Omega_L/\langle H\rangle} h(\phi[X]) = \sum_{\sigma\in\mathcal{S}_L}\frac{1}{1+e^{l(\sigma)}}.$$

This proves Theorem B.

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