A 4-dimensional Kleinian group.

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1. Introduction.

In this paper, we give an example of a 4-dimensional Kleinian group, i.e. a discrete group of isometries of hyperbolic 4-space, which is finitely generated but not finitely presented, and which is a subgroup of a cocompact Kleinian group.

Traditionally the term "Kleinian group" has been used to describe a discrete group, Γ, acting on hyperbolic 3-space, $\mathbf{H}^3$. In this dimension, there is a rich analytic theory, arising from the fact that the ideal boundary of $\mathbf{H}^3$ may be naturally identified with the Riemann sphere. An important class of results about such groups may be termed "finiteness theorems". Thus, under some mild hypothesis, typically that Γ be finitely generated, one deduces various other finiteness properties, which may be group theoretic, analytic, topological or geometric. We shall describe some basic finiteness theorems in a moment, but let's begin with some general observations. (More details can be found in [Bea].)

We shall write $\mathbf{H}^n$ for hyperbolic n-space, and $\operatorname{Isom}\mathbf{H}^n$ for the group of all isometries of $\mathbf{H}^n$. A group $\Gamma \subseteq \operatorname{Isom}\mathbf{H}^n$ is discrete, as a subgroup, if and only if it acts properly discontinuously on $\mathbf{H}^n$. In such a case, the quotient $\mathbf{H}^n/\Gamma$ is a hyperbolic orbifold. Moreover, a discrete group, Γ, is torsion-free if and only if it acts freely on $\mathbf{H}^n$. In this case, $\mathbf{H}^n/\Gamma$ is a hyperbolic manifold. The Selberg Lemma [Se] tells us that any finitely generated subgroup of $\operatorname{Isom}\mathbf{H}^n$ contains a torsion-free subgroup of finite index.

Hyperbolic space, $\mathbf{H}^n$, may be compactified by adjoining the *ideal sphere*, $\mathbf{H}^n_I$. This is natural, in the sense that the action of $\operatorname{Isom}\mathbf{H}^n$ on $\mathbf{H}^n$ extends to the compactification $\mathbf{H}^n \cup \mathbf{H}^n_I$. Moreover the action on $\mathbf{H}^n_I$ is conformal. If $\Gamma \subseteq \operatorname{Isom}\mathbf{H}^n$ is discrete, we may define the *limit set*, Λ, of Γ to be the set of accumulation points of some (any) Γ-orbit in $\mathbf{H}^n$. Thus $\Lambda \subseteq \mathbf{H}^n_I$ is closed, and we define the *discontinuity domain*, $\Omega = \mathbf{H}^n_I \backslash \Lambda$, so called because Γ acts properly discontinuously on Ω.

Returning to dimension 3, suppose $\Gamma \subseteq \operatorname{Isom}\mathbf{H}^3$ is finitely generated and discrete. Let $\Gamma' \subseteq \Gamma$ be a finite-index torsion-free subgroup. Thus $\mathbf{H}^3/\Gamma'$ is a hyperbolic 3-manifold, with $\pi_1(\mathbf{H}^3/\Gamma') \cong \Gamma'$ finitely generated. Now, Scott's theorem [Sc1] tells us that Γ' must be finitely presented. We conclude that Γ is finitely presented. This is one finiteness theorem. Another is Ahlfors's Finiteness Theorem [Ah]. This states that Ω/Γ (and hence Ω/Γ') is a (possibly disconnected) Riemann surface of finite type (i.e. a finitely-punctured compact surface). In particular, it is topologically finite. A theorem of Feighn and Mess [FMe] states that Γ has finitely many conjugacy classes of finite subgroups. Also, Sullivan's Cusp Finiteness Theorem [Su1] tells us that Γ has finitely many conjugacy classes of maximal parabolic subgroups. (A topological proof of this by Feighn and McCullough [FMc] also recovers the topological part of the conclusion of the Ahlfors Finiteness Theorem, except that it does not exclude the possibility of components of the quotient of the domain of discontinuity which are open discs. For further discussion of the Ahlfors Finiteness Theorem, see [KulS].) Finally, it is conjectured that the 3-manifold $\mathbf{H}^3/\Gamma'$ is "geometrically tame", as defined by Thurston [T]. This would imply, in particular, that $\mathbf{H}^3/\Gamma'$ is topologically finite (i.e. homeomorphic to the interior of a compact manifold). Bonahon [Bo] has proven the geometric tameness conjecture for a large class of groups, for example if Γ does not split as a free product.

For some time it was an open question as to what extent these results extend to higher dimensions. However, in a series of papers [KaP,Ka2,P1,P2], Kapovich and Potyagailo described counterexamples in dimension 4 to all the results stated above. In particular, in [KaP], they give an example of a finitely generated discrete torsion-free group $\Gamma \subseteq \mathrm{Isom}\,\mathbf{H}^4$ which does not admit a finite presentation, and for which the Ahlfors Finiteness Theorem fails in the strong sense that the fundamental groups of the components of Ω/Γ are not finitely generated.

In this paper we construct another such group which turns out to be a subgroup of a discrete cocompact group acting on $\mathbf{H}^4$. In particular it contains no parabolic elements—unlike Kapovich and Potyagailo's original example. Our example was inspired by theirs. (Note that Potyagailo [P2] has also described an example without parabolic elements.)

For other exotic 4-dimensional Kleinian groups of various sorts, see for example [ApT,BesC,Ka1,GLT,Kui]. (We are informed by the referee that [BesC] contained a gap which has been filled by [M].)

Note that some other consequences of the Ahlfors Finiteness Theorem remain unresolved for finitely generated groups in dimensions greater than or equal to 4; for example, if Ω_0 is a component of the discontinuity domain, does the limit set of the stabliser of Ω_0 necessarily coincide with $\partial\Omega_0$. It also seems to be unknown whether a finitely generated group with parabolics always admits a system of disjoint strictly invariant horoballs.

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2. Sketch.

As an example of the failure of Scott's Theorem in dimension 4, consider the following situation. Suppose M^3 is a 3-manifold with finitely generated fundamental group. Suppose that $S \subseteq M^3$ is a connected, incompressible, 2-sided, properly embedded surface of infinite topological type. By *incompressible* we mean that $\pi_1(S)$ injects into $\pi_1(M^3)$. Such pairs (M^3, S) certainly exist, as we shall see. (It does not concern us whether or not S separates M^3.)

Now, take two copies of M^3, and identify along the surface S, so as to obtain a complex D. More formally we may write $D = (M \times \{0,1\})/\!\sim$ where $(x,0) \sim (x,1)$ for all $x \in S$. Note that the fundamental group of D is an amalgamated free product:

$$\pi_1(D) \cong \pi_1(M^3) *_{\pi_1(S)} \pi_1(M^3).$$

We now want to embed D in a 4-manifold, M_1^4. Write I for the closed interval $[-1,1]$. Since S is 2-sided, we can find a regular neighbourhood $S \times I$ embedded in M^3 so that S is identified with $S \times \{0\}$. We now embed M as $M \times \{0\}$ in the 4-manifold $M \times I$. Thus $S \times I \times I$ is codimension-0 submanifold of $M \times I$. We now take two copies of $M \times I$, and identify along $S \times I \times I$, after a rotation of the square, $I \times I$, through an angle of $\pi/2$. This gives a 4-manifold M_1^4, in which D is properly embedded as a deformation retract (Figure 1). Thus $\pi_1(M_1^4) \cong \pi_1(D)$. Clearly $\pi_1(M_1^4)$ is finitely generated. The fact that it is not finitely presented follows from the following lemma, attributed to Neumann:

Lemma 2.1 : *Suppose A, B and C are groups with monomorphisms $\phi : C \longrightarrow A$ and $\psi : C \longrightarrow B$. If A and B are finitely generated, and the amalgamated free product $A *_C B$ is finitely presented, then C is finitely generated.*

Proof : Choose finite generating sets A_0 and B_0 for A and B respectively. If we identify A and B as subgroups of $A *_C B$, then $A_0 \cup B_0$ is a finite generating set for $A *_C B$. Let $\{r_1 \dots r_p\}$ be a complete set of relators corresponding to this generating set. Each relation $r_i = 1$ is a consequence of a finite number of relations in A and B, together with a finite number of relations of the form $\phi(c) = \psi(c)$ for $c \in C$. Let $C_0 \subseteq C$ be set of all $c \in C$ such that the relation $\phi(c) = \psi(c)$ occurs in some relation $r_i = 1$. Let C' be the subgroup of C generated by C_0. Then, the natural epimorphism from $A *_{C'} B$ to $A *_C B$ is an isomorphism. It follows that $C' = C$, and so C is finitely generated. $\diamondsuit$

(In fact, in the case in which we are interested, $H_1(C, \mathbf{Z})$ has infinite rank, and so the Mayer-Vietoris sequence for $A *_C B$ tells us that $H_2(A *_C B, \mathbf{Z})$ also has infinite rank.)

We next give a sketch of how we intend to realise this example geometrically. A more rigorous treatment will be given in the context of an explicit example in Section 4.

Suppose that $M^3 = \mathbf{H}^3/\Gamma_3$ is a hyperbolic 3-manifold with finitely generated fundamental group $\pi_1(M^3) \cong \Gamma_3$. Suppose that $S \subseteq M^3$ is a properly embedded totally geodesic 2-sided surface of infinite topological type. Suppose, moreover, that S has a uniform regular neighbourhood in M^3, i.e. that for some sufficiently large $r > 0$, the metric r-neighbourhood, $N_r(S)$ of S is topologically a product $S \times I$. (It turns out that $r = \cosh^{-1} \sqrt{2}$ will do.)

Given $h > 0$, we may realise $M^3 \times I$ as a hyperbolic 4-manifold with convex boundary as follows. We identify $\mathbf{H}^3$ as a totally geodesic subspace σ of $\mathbf{H}^4$, and we extend the action of Γ_3 to $\mathbf{H}^4$. Thus the uniform neighbourhood, $N_h(\sigma)$ is Γ_3-invariant, and so we may form the quotient $N_h(\sigma)/\Gamma_3 \equiv M^3 \times I$. Now, if $r > h$, we may take two copies of $M^3 \times I$ and superimpose them so that the two copies if $M^3 \equiv M^3 \times \{0\}$ sitting inside overlap orthogonally along S. Identifying the superimposed pieces of $M^3 \times I$ we arrive, as before with a 4-manifold with boundary, M_1^4. The boundary of M_1^4 will not be convex. However, if r is sufficiently large, and $h < r$ is chosen appropriately, we can smooth out the boundary locally so that it becomes convex. It suffices to verify this in a 2-dimensional cross-section (Figure 2). In fact, this is an example of a more general construction due to Thurston, of which we shall give a more careful account in Section 5. Another general construction (Lemma 5.2) allows us to embed the 4-manifold thus obtained, as a deformation retract, inside a compete hyperbolic 4-manifold without boundary, M^4. The group of covering transformations of M^4 is thus a finitely generated non-finitely presented Kleinian group.

It's not hard to see that this group also gives a counterexample to the topological part of the Ahlfors Finiteness Theorem in dimension 4. Note that the quotient of the discontinuity domain is homeomorphic to the boundary of M_1^4. This boundary has either four or one components, depending on whether or not S separates M^3. If S does not separate, then the single boundary component has infinitely generated fundamental group. If S does separate, then two of the boundary components have infinitely generated fundamental group.

Consider the case where S separates M^3 into two pieces, N_1^3 and N_2^3, each with boundary S. Now two of the boundary components of M_1^4 are homeomorphic to M^3, while the other two are homeomorphic respectively to DN_1^3 and DN_2^3, where DN_i^3 is obtained by doubling N_i^3 in its boundary. To see that DN_i^3 has infinitely generated fundamental group, set $A = \pi_1(N_i^3)$ and $C = \pi_1(S)$; so we may regard C as a subgroup of A, and $\pi_1(DN_i^3) = A *_C A$. There is a natural epimorphism from $A *_C A$ to A, and so if $A *_C A$ is finitely generated, then so is A. By Lemma 2.1, $A *_C A$ is not finitely presented. But $A *_C A$ is a 3-manifold group, and so this contradicts Scott's theorem. A similar argument deals with the case where S does not separate M^3.

We now describe how to find a suitable pair (M^3, S). Choose a compact hyperbolic 3-manifold, M_1^3, which fibres over the circle, and which contains an immersed closed totally geodesic surface, $S_1 \longrightarrow M^3$. We choose an immersion which does not factor through a covering $S_1 \longrightarrow S_2$. An example of such will be described below. We may now find a finite covering, M_0^3, of M_1^3, which contains a totally geodesic surface S_0. Moreover, we may assume that S_0 has an arbitrarily wide uniform regular neighbourhood in M_0^3. These statements follow from a result of Long [L], which imply that the image, H, of $\pi_1(S_1)$ in $\pi_1(M_0^3)$ is separable. (In fact, Long states that there is a subgroup, H', of G which is separable in G, and which contains H as a subgroup of finite index; this index is one because we chose the immersion $S_1 \longrightarrow M^3$ so as not to factor through a covering.) However, in the example we shall describe, these constructions can be made explicit (see Lemma 3.5). After passing to finite covers if necessary, we can assume that both M_0^3 and S_0 are orientable, so S_0 is 2-sided in M_0^3.

Now, let M^3 be the infinite cyclic covering of M_0^3 corresponding to the fibre subgroup of $\pi_1(M_0^3)$. Let S be a component of the inverse image of S_0 under the covering projection. Thus S is a covering space of S_0. This covering must be either infinite cyclic or trivial. However, the latter case is clearly impossible, since it would mean that the fibre subgroup would be fuchsian, whereas its quotient, M^3, is geometrically infinite. We conclude that S has infinite topological type. The uniform regular neighbourhood about S_0 in M_0^3 lifts to one about S in M^3.

As an explicit example, we use the following construction of Thurston [Su2]. First note that we may represent the dodecahedron combinatorially as a cube with six edges added in the pattern shown in Figure 3. If we identify opposite faces of the cube so as to form a 3-torus, then these additional edges become three disjoint embedded circles. We define a 3-orbifold by assigning to each of these circles a transverse cone angle equal to π. Note that this orbifold fibres over the circle (given by the long diagonal of the cube).

Now, this orbifold has a hyperbolic structure formed by realising the dodecahedron as a right regular dodecahedron in $\mathbf{H}^3$. By "right" we mean that all the dihedral angles are equal to $\pi/2$. Let $M_2^3 = \mathbf{H}^3/\Gamma_2$ be the hyperbolic orbifold thus obtained. Note that Γ_2 is commensurable with the group, G_3, generated by the reflections in the faces of a right regular dodecahedron. In fact Γ_2 and G_3 are both finite-index subgroups of the (tetrahedral) group of symmetries of a right regular dodecahedral tessellation of $\mathbf{H}^3$.

Now, by the Selberg Lemma, we know that Γ_2 contains a torsion free subgroup of finite index, Γ_1, which we can suppose is a subgroup also of G_3. Thus, $M_1^3 = \mathbf{H}^3/\Gamma_1$ is a compact hyperbolic 3-manifold fibring over the circle, and tiled by dodecahedra. It clearly contains an immersed totally geodesic surface, S_1, formed as a union of pentagonal faces. We now lift to obtain the pair (M^3, S) as described above.

We may now obtain a complete hyperbolic 4-manifold, M^4, using two copies of M^3, in the manner described earlier. Since M^3 is tiled by dodecahedra, we see that M^4 will be tiled by right regular 120-cells. (Recall that a 120-cell is the regular 4-dimensional polyhedron with 120 dodecahedral faces. It may be realised as a compact hyperbolic polyhedron with all dihedral angles equal to $\pi/2$ which is made up of 14400 fundamental domains for the Coxeter group $\circ \overset{5}{—} \circ — \circ — \circ \overset{4}{—} \circ$.) Let $\Gamma \subseteq \operatorname{Isom} \mathbf{H}^4$ be the group of covering transformation of M^4, so that $M^4 = \mathbf{H}^4/\Gamma$. We see that G is a subgroup of the group $G_4 \subseteq \operatorname{Isom} \mathbf{H}^4$, generated by reflections in the faces of a right regular 120-cell. In summary, we have that Γ is a finitely generated non-finitely presented subgroup of the discrete cocompact group G_4.

(Note that Davis [D] describes a compact hyperbolic 4-manifold built out of 120-cells, though in that case, the link of each two dimensional face is a pentagon, rather than a square.)

We have described all the essential ingredients of our example, though we made appeal to some general principles which were not clearly elucidated. To give a more rigorous treatment, we make some observations about certain tessellations of hyperbolic space.

3. Right tessellations.

In this section, we describe tessellations obtained by continually reflecting a right-polyhedron in its codimension-1 faces. Note that it follows from the work of Vinberg and Nikulin that such right-angled polyhedra can exist in $\mathbf{H}^n$ only for $n \leq 4$. (See for example [N]). In fact any convex 5-dimensional polyhedron must contain a 2-dimensional face with at most four edges.

Definition : A *right polyhedron* P in $\mathbf{H}^n$ is a compact convex polyhedron with non-empty interior, such that all the dihedral angles are equal to $\pi/2$.

By a *face* of P we mean the intersection of P with a supporting hyperplane. Note that each face of P is itself a right polyhedron of lower dimension. We write $\mathcal{F}(P)$ for the set of all codimension-1 faces of P. Thus, $\partial P = \bigcup \mathcal{F}(P)$. We say that $F_1, F_2 \in \mathcal{F}(P)$ are *adjacent* if $F_1 \cap F_2 \neq \emptyset$. In such a case, $F_1 \cap F_2$ will be a codimension-2 face of P. This follows from the fact that the link of every vertex of P is an $(n-1)$-simplex. (Note that such a link is a convex polyhedron in the $(n-1)$-sphere, all of whose dihedral angles are equal to $\pi/2$, and so the vertices of the dual form an orthonormal basis for $\mathbf{R}^n$.)

Lemma 3.1 : *Suppose P is a right polyhedron in $\mathbf{H}^n$, and $F_1, F_2 \in \mathcal{F}(P)$. Let σ_1 and σ_2 be the codimension-1 subspaces of $\mathbf{H}^n$ containing F_1 and F_2 respectively. If $F_1 \cap F_2 = \emptyset$, then $\sigma_1 \cap \sigma_2 = \emptyset$.*

Proof : Let α be the shortest geodesic from F_1 to F_2. Since P is convex, $\alpha \subseteq P$. Since P is a right polyhedron, we see that α meets σ_1 and σ_2 orthogonally. Thus α is the shortest geodesic from σ_1 and σ_2. $\diamondsuit$

By a *4-chain* in $\mathcal{F}(P)$, we mean a cyclically ordered set of four distinct elements, $\{F_1, F_2, F_3, F_4\}$, of $\mathcal{F}(P)$, such that $F_i \cap F_{i+1} \neq \emptyset$ and $F_i \cap F_{i+2} = \emptyset$ for $i = 1, 2, 3, 4$, where subscripts are taken mod 4.

Lemma 3.2 : *If P is a right polyhedron, then $\mathcal{F}(P)$ contains no 4-chain.*

Proof : Suppose F_1, F_2, F_3, F_4 is a 4-chain. Let σ_i be the codimension-1 subspace spanned by F_i. We know that σ_i meets σ_{i+1} orthogonally, and by Lemma 3.1, that $\sigma_i \cap \sigma_{i+2} = \emptyset$. It's easy to see that this is impossible. $\diamondsuit$

We say that a set of codimension-1 faces, $\mathcal{F}_0 \subseteq \mathcal{F}(P)$ are *mutually adjacent* if $F_1 \cap F_2 \neq \emptyset$ for all $F_1, F_2 \in \mathcal{F}_0$.

Lemma 3.3 : *If $\mathcal{F}_0 \subseteq \mathcal{F}(P)$ is a set of mutually adjacent faces, then $\bigcap \mathcal{F}_0 \neq \emptyset$. In fact, $\bigcap \mathcal{F}_0$ is a codimension-r face of P, where $r = |\mathcal{F}_0|$ (so that $|\mathcal{F}_0| \leq n$).*

Proof : Choose any $F_0 \in \mathcal{F}_0$, so that F_0 is an $(n-1)$-dimensional right polyhedron. Let $\mathcal{F}_0' = \{F \cap F_0 \mid F \in \mathcal{F}_0 \setminus \{F_0\}\}$. Thus, $\mathcal{F}_0'$ is a set of $r-1$ codimension-1 faces of F_0. Applying Lemma 3.1 to F_0, we find that these faces are mutually adjacent. By induction on dimension, we conclude that $r \leq n$, and that $\bigcap \mathcal{F}_0 = \bigcap \mathcal{F}_0'$ is an $(n-r)$-dimensional face of F_0, and hence of P. $\diamondsuit$

Lemma 3.4 : *Suppose $\mathcal{F}_0 \subseteq \mathcal{F}(P)$ is a set of mutually adjacent faces. Let $\mathcal{F}_1$ be the set of faces of $\mathcal{F}(P) \setminus \mathcal{F}_0$ which are adjacent to some element of $\mathcal{F}_0$. If $F_1, F_2 \in \mathcal{F}_1$ are adjacent, then there is some $F_0 \in \mathcal{F}_0$ adjacent to both.*

Proof : Suppose, for contradiction, that there is no such F_0. By hypothesis, there are elements $F_3, F_4 \in \mathcal{F}_0$ with F_3 adjacent to F_2, and with F_4 adjacent to F_1. We must have $F_1 \cap F_3 = \emptyset$ and $F_2 \cap F_4 = \emptyset$. Thus $F_3 \neq F_4$, and so F_3 and F_4 are adjacent. It follows that F_1, F_2, F_3, F_4 is a 4-chain, contradicting Lemma 3.2. $\diamondsuit$

Definition : A *right tessellation* of $\mathbf{H}^n$ is a collection, $\mathcal{P}$, of n-dimensional right polyhedra which tessellate $\mathbf{H}^n$ (i.e. the interiors are disjoint, and $\bigcup \mathcal{P} = \mathbf{H}^n$), and such that if any two elements of $\mathcal{P}$ intersect, then they do so in a common face.

Another way of describing right tessellations is as follows. Suppose $\mathcal{S}$ is a locally finite collection of codimension-1 subspaces of $\mathbf{H}^n$, with the property that any two elements of $\mathcal{S}$ intersect orthogonally or not at all. Suppose that each component of $\mathbf{H}^n \setminus \bigcup \mathcal{S}$ is relatively compact. Then the set, $\mathcal{P}$, of closures of these components is a right tessellation of $\mathbf{H}^n$. Moreover, every right tessellation arises in this way. We write $\mathcal{S} = \mathcal{S}(\mathcal{P})$. Note that $\bigcup \mathcal{S}(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} \mathcal{F}(P)$. In fact, if $P \in \mathcal{P}$, then $\mathcal{F}(P) = \{S \cap P \mid S \in \mathcal{S}\}$. Also, by Lemma 3.1, we see that if $S_1, S_2 \in \mathcal{S}$ and $P \in \mathcal{P}$ meet pairwise (i.e. $P \cap S_1$, $P \cap S_2$ and $S_1 \cap S_2$ are all non-empty), then $P \cap S_1 \cap S_2 \neq \emptyset$.

Definition : We say that a subset $\mathcal{S}_0 \subseteq \mathcal{S}$ is *sparse* if whenever $S_1, S_2 \in \mathcal{S}$ and $P \in \mathcal{P}$ satisfy $P \cap S_1 \neq \emptyset$ and $P \cap S_2 \neq \emptyset$, then $S_1 \cap S_2 \neq \emptyset$ (and so $P \cap S_1 \cap S_2 \neq \emptyset$). In other words, no polyhedron of $\mathcal{P}$ can meet two disjoint elements of $\mathcal{S}_0$.

Lemma 3.5 : *Suppose $\mathcal{P}$ is a right tessellation, and that $\mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{P})$ is sparse. Suppose that $\bigcup \mathcal{S}_0$ is connected. Let $\mathcal{P}_0 = \{P \in \mathcal{P} \mid P \cap S \neq \emptyset \text{ for some } S \in \mathcal{S}_0\}$. Then $\bigcup \mathcal{P}_0$ is convex.*

Proof : Let $\Sigma = \bigcup \mathcal{S}_0$ and $\Pi = \bigcup \mathcal{P}_0$. Given $P \in \mathcal{P}_0$, we write $\mathcal{F}(P)$ as a disjoint union $\mathcal{F}(P) = \mathcal{F}_0(P) \sqcup \mathcal{F}_1(P) \sqcup \mathcal{F}_2(P)$, where $\mathcal{F}_0(P) = \{F \in \mathcal{F}(P) \mid F \subseteq \Sigma\}$, $\mathcal{F}_1(P) = \{F \in \mathcal{F}(P) \mid F \cap \Sigma \neq \emptyset, F \not\subseteq \Sigma\}$ and $\mathcal{F}_2(P) = \{F \in \mathcal{F}(P) \mid F \cap \Sigma = \emptyset\}$. Note that, since Σ_0 is sparse, the faces $\mathcal{F}_0(P)$ are mutually adjacent, and so $\mathcal{F}_0(P)$ and $\mathcal{F}_1(P)$ satisfy the hypotheses of Lemma 3.4. Thus, any two adjacent faces in $\mathcal{F}_1(P)$ have a common adjacent face in $\mathcal{F}_0(P)$.

Let $\mathcal{F}_0 = \bigcup_{P \in \mathcal{P}_0} \mathcal{F}_0(P)$ and $\mathcal{F}_1 = \bigcup_{P \in \mathcal{P}_0} \mathcal{F}_1(P)$. It is easy to see that if F is any element of $\mathcal{F}_0 \cup \mathcal{F}_1$, then the two polyhedra of $\mathcal{P}$ which have F as a face both lie in $\mathcal{P}_0$. We see that $\Pi_I = \Pi \setminus \bigcup_{P \in \mathcal{P}_0} \mathcal{F}_2(P)$ is an open neighbourhood of Σ in $\mathbf{H}^n$. Let Π_C be the metric completion of Π_I in the induced path-metric. Thus, Π_C is a manifold with boundary, and there is a finite-to-one map $p : \Pi_C \longrightarrow \Pi$. We claim that Π_C has convex boundary. It then follows that p is injective. Since Σ is connected, we then see that $\Pi \equiv \Pi_C$ is convex.

We may construct, abstractly, the manifold Π_C by gluing together the polyhedra of $\mathcal{P}_0$ along the faces $\mathcal{F}_0 \cup \mathcal{F}_1$. The boundary, $\partial \Pi_C$, is tiled by the elements of $\mathcal{F}_2 = \bigsqcup_{P \in \mathcal{P}_0} \mathcal{F}_2(P)$, considered as a disjoint union. (It is conceivable, for the moment, that there may be distinct elements $P, P' \in \mathcal{P}_0$ for which $\mathcal{F}_2(P) \cap \mathcal{F}_2(P') \neq \emptyset$.)

Suppose that $F, F' \in \mathcal{F}_2$ meet along an $(n-2)$-dimensional face $K \subseteq \partial \Pi_C$. A priori, the interior angle at which F and F' meet may be $\pi/2$, π, $3\pi/2$ or 2π, according to whether 1, 2, 3 or 4 polyhedra in $\mathcal{P}_0$ have K as a face (Figure 4). To see that Π_C has convex boundary, we want to rule out the latter two cases.

However, in the latter two cases, we see that there is some polyhedron $P \in \mathcal{P}_0$, and faces $F_1, F_2 \in \mathcal{F}(P)$ distinct from F and F', with $F_1 \cap F_2 = K$. Since $K \cap \Sigma = \emptyset$, we must have $F_1, F_2 \in \mathcal{F}_1(P)$. Thus, there is some $F_0 \in \mathcal{F}_0(P)$ with F_0, F_1 and F_2 mutually adjacent. By Lemma 3.3, we have $F_0 \cap F_1 \cap F_2 \neq \emptyset$. But $F_0 \subseteq \Sigma$, and so $K \cap \Sigma \neq \emptyset$. This contradiction shows that only the first two cases can occur, and so Π_C has convex boundary. $\diamondsuit$

This is probably not the most elegant proof one could give of this lemma. However, it is in a form that admits a modification to the situation which really interests us, namely when we are given the complex Σ abstractly, together with a tiling of Σ by right regular polyhedra (dodecahedra). We may then use the combinatorial structure to construct Π_C out of regular polyhedra one dimension higher (120-cells). The argument shows that Π_C has convex boundary, and thus embeds as a convex subset of hyperbolic space ($\mathbf{H}^4$). We describe this more carefully in Section 4.

Suppose, now that $P \subseteq \mathbf{H}^n$ is a right polyhedron. Let G be the group generated by reflections in the faces $\mathcal{F}(P)$. Then $G \subseteq \operatorname{Isom} \mathbf{H}^n$ is discrete and cocompact, and $\mathcal{P} = GP = \{\gamma P \mid \gamma \in G\}$ is a G-invariant right tessellation of $\mathbf{H}^n$. Right tessellations arising in this way are characterised by the fact that $\mathcal{P}$ is invariant under reflection in S for all $S \in \mathcal{S}(\mathcal{P})$.

The Selberg Lemma tells us that G contains a torsion free subgroup of finite index. However, in this situation, there is an elementary geometric construction of such a subgroup as follows. We choose an m-colouring of the codimension-1 faces of P, i.e. a map $c : \mathcal{F}(P) \longrightarrow \{1, \dots, m\}$ such that no two adjacent faces are given the same colour. Let $\mathcal{F} = \bigcup_{P \in \mathcal{P}} \mathcal{F}(P) = \{gF \mid g \in G, F \in \mathcal{F}(P)\}$. We may extend c to a map $c : \mathcal{F} \longrightarrow \{1, \dots, m\}$ by setting $c(gF) = c(F)$ for $g \in G$. It is not hard to see that this map is well defined. Given $i \in \{1, \dots, m\}$, let $\Sigma(i) = \bigcup\{F \in \mathcal{F} \mid c(F) = i\}$. We see that $\Sigma(i)$ has the form $\bigcup \mathcal{S}(i)$, where $\mathcal{S}(i) \subseteq \mathcal{S}$ is a G-invariant collection of disjoint codimension-1 subspaces.

Now, we may write any $g \in G$ as a product $r(F_1)r(F_2)\dots r(F_k)$ where $r(F)$ is reflection in the face $F \in \mathcal{F}(P)$. Let $\rho(g) = (\epsilon_1, \epsilon_2, \dots, \epsilon_m)$, where $\epsilon_i \in \mathbf{Z}_2$ is the number of times, mod 2, that a face of colour i occurs in this product. We see that $\rho(g)$ is well defined, and that the map ρ gives a homomorphism from G onto $\mathbf{Z}_2^m = \mathbf{Z}_2 \times \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$. (For another description of $\rho(g)$, choose an interior point, x, of P, and join x to gx by a general position path β. The ith entry of $\rho(g)$ is then the number of times, mod 2, that β intersects $\Sigma(i)$.) Let $G' = \ker \rho$. We see that G' preserves orientation, and has index 2^m in G. Moreover, if $g \in G'$ and $P' \in \mathcal{P}$, then $P' \cap gP' = \emptyset$. We conclude easily that G' is torsion free. Thus, $\mathbf{H}^n/G'$ is a compact orientable manifold, tiled by embedded polyhedra projected from $\mathcal{P}$. Also, for any $i \in \{1, \dots, m\}$, we see that $S(i) = \Sigma(i)$ is an embedded totally geodesic codimension-1 submanifold.

We remark that we may compose ρ with the homomorphism $q : \mathbf{Z}_2^m \longrightarrow \mathbf{Z}_2^{m-1}$ given by quotienting out the diagonal. In this way we get a homomorphism $q \circ \rho : G \longrightarrow \mathbf{Z}_2^{m-1}$. The kernel of this homomorphism is again torsion-free, and of index 2^{m-1}. Note that if the dimension n is 3, then the Four Colour Theorem gives us a torsion free subgroup of index 8 in G, which is clearly the best possible. This argument can be found in [V]. It was shown to us by Tadeusz Januszkiewicz.

Suppose now, that there is precisely one face in $\mathcal{F}(P)$ of any given colour i. Then, the totally geodesic submanifold $S(i) \subseteq \mathbf{H}^n/G'$ is connected. Moreover, the subset $\mathcal{S}(i) \subseteq \mathcal{S}$ is sparse, i.e. no polyhedron of $\mathcal{P}$ meets two distinct elements of $\mathcal{S}(i)$. In this case we say that $\mathcal{S}(i)$ is *collared*. More formally:

Definition : Suppose $G_0 \subseteq G$ is torsion free, so that $\mathbf{H}^n/G_0$ is a manifold tiled by polyhedra. (We can suppose, if we like, that these polyhedra are embedded, though this is not essential.) By a *collared* codimension-1 submanifold, $S_0 \subseteq \mathbf{H}^n/G_0$, we mean an embedded 2-sided totally geodesic submanifold, composed of codimension-1 faces of the tiling of $\mathbf{H}^n/G_0$, and such that $S_0 = \bigcup \mathcal{S}_0$, where $\mathcal{S}_0 \subseteq \mathcal{S}$ is sparse.

Lemma 3.6 : *Suppose that $G \subseteq \mathrm{Isom}\,\mathbf{H}^n$ is generated by reflections in the codimension-1 faces of a right polyhedron P. Let $\mathcal{P} = GP$ be the resulting right tessellation. If $G_1 \subseteq G$ is a finite index subgroup, then there is a finite index torsion-free orientation preserving subgroup $\Gamma_0 \subseteq G_1$, so that the quotient manifold $\mathbf{H}^n/\Gamma_0$ contains a collared totally geodesic codimension-1 submanifold, $S_0 \subseteq \mathbf{H}^n/\Gamma_0$.*

Proof : Colour the faces $\mathcal{F}(P)$ so that no two have the same colour. Let $G' \subseteq G$ be the resulting subgroup as described above, and let $\Sigma(1)$ be as described. Let $\Gamma_0 = G_1 \cap G'$, and let S_0 be a connected component of $\Sigma(1)/\Gamma_0$. $\diamondsuit$

We remark that there is an abundance of finite index subgroups G, which can be constructed geometrically, following the ideas in [Sc2]. For example residual finiteness of G follows from the fact that any two polyhedra of $\mathcal{P}$ are contained in a convex set which is a finite union of polyhedra of $\mathcal{P}$. Also all geometrically finite subgroups of G are separable.

4. An explicit example.

Let $G_3 \subseteq \mathrm{Isom}\,\mathbf{H}^3$ be the discrete group of isometries generated by reflections in the faces of a right regular dodecahedron. In Section 2, we described a commensurable group $\Gamma_2 \subseteq \mathrm{Isom}\,\mathbf{H}^3$ so that $\mathbf{H}^3/\Gamma_2$ is a compact orbifold fibring over the circle. Applying Lemma 3.6, we obtain a subgroup $\Gamma_0 \subseteq \Gamma_2 \cap G_3$ so that $M_0^3 = \mathbf{H}^3/\Gamma_0$ is a compact orientable manifold containing a connected orientable collared totally geodesic surface $S_0 \subseteq M_0^3$ which is a union of pentagonal faces. Now M_0^3 also fibres over the circle, so we may form the infinite cyclic cover M^3 of M_0^3. Thus, M^3 is tiled by dodecahedra, and contains a connected collared surface $S \subseteq M^3$ tiled by pentagons, and of infinite topological type. We form a complex D by joining together two copies of M^3 along S.

Now, let $G_4 \subseteq \mathrm{Isom}\,\mathbf{H}^4$ be the discrete subgroup generated by reflections in the faces of a right regular 120-cell, P. Let $\mathcal{P} = G_4P$ be the resulting tessellation of $\mathbf{H}^4$, and let $\mathcal{S} = \mathcal{S}(\mathcal{P})$. Note that we may identify the setwise stabliser of any $S \in \mathcal{S}$ with G_3.

Let Σ be the universal cover of the complex D. Thus, Σ consists of a locally finite countable union of hyperbolic 3-spaces glued together along disjoint planes. If we imagine these 3-spaces as meeting orthogonally, then there is a natural way of developing Σ into the 3-skeleton, $\bigcup \mathcal{S}$, of the tessellation $\mathcal{P}$. We claim that the developing map is injective. To see this, we apply the argument of Lemma 3.5. Using the combinatorial structure of the tiling of Σ, we construct abstractly the simply connected manifold Π_C out of 120-cells.

As in Lemma 3.5, we see that Π_C embeds as a convex subset of $\mathbf{H}^4$. In fact it has the form $\Pi_C = \bigcup \mathcal{P}_0$ for some subset $\mathcal{P}_0 \subseteq \mathcal{P}$. From the naturality of the construction, the action of $\pi_1(D)$ on Σ extends to Π_C, and hence to $\mathbf{H}^4$ as a subgroup of G_4. This subgroup is finitely generated but does not admit a finite presentation.

5. Appendix.

The purpose of this section is to give an account of the construction of Thurston mentioned in Section 2. This construction is a generalisation of "bending". It was used by Thurston to construct examples of distinct geometrically finite representations of the same group into $\mathrm{Isom}\,\mathbf{H}^3$ which are not quasiconformally conjugate; see [T, page 9.52].

Suppose V is a hyperbolic n-manifold (not necessarily connected) with totally geodesic boundary consisting of finitely many components $F_1, \ldots, F_k$, which we take to be cyclically ordered. Suppose that $F_1, \ldots, F_k$ are all isometric. We may form a metric complex, D, by identifying all these components by isometry to give an $(n-1)$-manifold $F \subseteq D$. (Thus $D \setminus F$ may be identified with the interior of V.) At each point $x \in F$, there are well defined tangent vectors $\xi_1(x), \ldots, \xi_k(x)$ to D, perpendicular to F, in natural bijective correspondence to the boundary components of V.

The aim of the construction is to give an embedding $\iota : D \hookrightarrow W$ of D in a complete hyperbolic $(n+1)$-manifold W, without boundary, such that

(1) The metric on D agrees with the path metric induced from W.

(2) Each component of $\iota(D \setminus F)$ is totally geodesic in W.

(3) W retracts onto D.

We also want to be able to specify the angles, θ_i, at which the components of $\iota(D \setminus F)$ meet along F. In other words, given numbers $\theta_i \in (0, 2\pi)$ for $i = 1, \ldots, k$ summing to 2π, we want to arrange that for some (and hence every) $x \in F$, the vectors $\iota_* \xi_i(x)$ and $\iota_* \xi_{i+1}(x)$ meet at an angle of θ_i, taking account of orientation and cyclic ordering (Figure 5). We show that we can always find such an embedding provided that there are large enough collars around each the boundary components F_i.

Proposition 5.1 : *There is a map $r : (0, \pi) \longrightarrow (0, \infty)$ such that the following holds. Suppose $V, F_1, \ldots, F_k, D, F$ are as described above. Suppose that $\theta_1, \ldots, \theta_k \in (0, 2\pi)$ are such that $\sum_{i=1}^k \theta_i = 2\pi$. Let $\theta = \min\{\theta_i \mid 1 \leq i \leq k\}$ and $r = r(\theta)$. Suppose that, for each $i \in \{1, \ldots, k\}$, the uniform neighbourhood $N_r(F_i)$ is an embedded collar (i.e. it retacts onto F_i). Then, there is an embedding $\iota : D \hookrightarrow W$ of D in a complete hyperbolic $(n+1)$-manifold, W, (without boundary), satisfying (1), (2) and (3) above, and for which the quantities $\theta_1, \ldots, \theta_k$ measure the angles at which the collars $\iota(N_r(F_i))$ meet along F (in the sense described above). Moreover, the pair (W, D) is unique up to isometry.*

In fact, we may take $r(\theta) = \cosh^{-1} \mathrm{cosec}(\theta/2)$. Note that $r(\theta) \to \infty$ as $\theta \to 0$, and $r(\theta) \to 0$ as $\theta \to \pi$.

We have already seen one example of this construction, namely gluing together two copies of a hyperbolic 3-manifold M^3 along a totally geodesic surface $S \subseteq M^3$. In this case, we regard $M^3 \backslash S$ as a path-metric space—distances are given by the infinum of length of paths, and if S separates, the components of $M \backslash S$ have infinite distance from each other. We take V is the metric completion of two copies of $M^3 \backslash S$.

Another example, mentioned above, is the bending of an n-manifold, M, along a totally geodesic codimension-1 submanifold, S. Let V be the metric completion of $M \backslash S$ in the induced path-metric. Thus V has boundary components F_1 and F_2 isometric to S. Given $\phi \in (0, \pi)$, let $\theta_1 = \pi - \phi$ and $\theta_2 = \pi + \phi$. In this case, the construction describes *bending* through an angle of ϕ. If S admits a 2-sided collar (for example if S is compact), then we can always bend through some positive angle (since $r(\theta) \to 0$ as $\theta \to \pi$). If S admits a very large 2-sided collar (for example if M is a 2-manifold, and S is a short simple closed curve), then we can bend through an angle very close to π.

The uniqueness of the manifold W is fairly clear. Note that the universal cover $\tilde{D}$ of D is an embedded complex in $\mathbf{H}^{n+1}$. This embedding is determined, up to isometry in $\mathbf{H}^n$, by the metric structure of $\tilde{D}$ and the angles θ_i. Thus, the action of $\pi_1(D)$ on $\mathbf{H}^n$ is determined, and the quotient, W, is unique.

We need to show the existence of W. Note that it suffices to find an $(n+1)$-manifold, W', with convex boundary satisfying the same properties. This is because:

Lemma 5.2 : *Every complete hyperbolic n-manifold M' with convex boundary embeds in a complete hyperbolic n–manifold M without boundary. The pair (M, M') is unique up to isometry.*

Proof : We may develop the universal cover of M' into $\mathbf{H}^n$. Since this is connected and has convex boundary this must be an embedding. Extend the action of $\pi_1(M')$ to $\mathbf{H}^n$ and take the quotient. $\diamondsuit$

We shall need the following construction. Suppose that $\sigma \subseteq \mathbf{H}^{q+r}$ is a subspace of dimension q. There is a natural map $f : \mathbf{H}^{q+r} \longrightarrow \mathbf{H}^r$ such that σ gets mapped to a single point $x_0 \in \mathbf{H}^r$, such that each r-dimensional subspace orthogonal to σ gets mapped isometrically to $\mathbf{H}^r$, and such that for any subspace μ of $\mathbf{H}^r$ containing x_0, $f^{-1}\mu$ is a subspace of $\mathbf{H}^{q+r}$. If $K \subseteq \mathbf{H}^r$ and $X \subseteq \sigma$, then we have a well defined subset $Y = Y(X, K) \subseteq \mathbf{H}^{q+r}$ and such that if τ is an r-dimensional subspace of $\mathbf{H}^{q+r}$ orthogonal to σ then either $Y \cap \tau = \emptyset$ and $X \cap \tau = \emptyset$, or $f(Y \cap \tau) = K$ and $X \cap \tau \neq \emptyset$. There is a natural projection $\tilde{p} : Y \longrightarrow X$.

Now suppose that $M = \mathbf{H}^q / \Gamma$ is a hyperbolic q-manifold with convex boundary, and that $K \subseteq \mathbf{H}^r$. We identify $\mathbf{H}^q$ with σ so that the universal cover $\tilde{M}$ becomes a convex subset of σ. The action of Γ extends to $\mathbf{H}^{q+r}$, so we may define $W(M, K) = Y(\tilde{M}, K)/\Gamma$. Let $p : W(M, K) \longrightarrow M$ be the projection induced by $\tilde{p} : Y(\tilde{M}, K) \longrightarrow \tilde{M}$. Note that if $J \subseteq K$, then there is a natural embedding $W(M, J) \subseteq W(M, K)$.

Proof of Proposition 5.1 : We are given $V, F_1, \ldots, F_k, D, F, \theta_1, \ldots, \theta_k$. We want to construct W. The idea is to construct W' by gluing together two $(n+1)$-manifolds with boundary. One, W_0, is homeomorphic to F times a disc, and the other, W_1, is homeomorphic to V times an interval.

Fix $x_0 \in \mathbf{H}^2$. Let $\beta_1, \dots, \beta_k$ be geodesic rays in $\mathbf{H}^2$, based at x_0, so that the angle between β_i and β_{i+1} is θ_i. Choose some $h > 0$, and let $A = N_h(\bigcup_{i=1}^k \beta_i)$. Let H be the convex hull of A. We see that H has k ends going off to infinity, each corresponding to a ray β_i. Suppose $r > 0$. Let y_i be the point on the ray β_i such that $d(x_0, y_i) = r$ and let J_i be the geodesic segment of length $2h$ centred on y_i and orthogonal to β_i. If r is sufficiently large, depending on h and θ, then each such segment J_i will separate the end of H corresponding to β_i. In this case we may cut off all the ends of H along $\bigcup_{i=1}^k J_i$ to leave a compact set $K \subseteq H$, with $\bigcup_{i=1}^k J_i \subseteq \partial K$ (Figure 6). Note that the pair $(K, \bigcup_{i=1}^k J_i)$ retracts onto $(\bigcup_{i=1}^k \beta_i \cap N_r(x_0), \{y_1, \dots, y_k\})$. For any fixed θ, the best value of r is obtained by letting $h \to \infty$. Simple hyperbolic trigonometry shows that this gives $r(\theta) = \cosh^{-1} \operatorname{cosec}(\theta/2)$.

Now, let $W_0 = W(F, K)$. For each $i \in \{1, \dots, k\}$, let $T_i = W(F, J_i) \subseteq \partial W_0$.

Now choose $x_0' \in \mathbf{H}^1$, and let I be the closed interval of length $2h$ centred on x_0'. This gives us an $(n+1)$-manifold $W(V, I)$, and a projection $p : W(V, I) \longrightarrow V$. Let $W_1 = p^{-1}(V \setminus \bigcup_{i=1}^k \operatorname{int} N_r(F_i))$, and let $T_i' = p^{-1}(\partial N_r(F_i)) \subseteq \partial W_1$. We see that there is a natural isometry from T_i to T_i'. We form our manifold W' by taking a disjoint union $W_0 \sqcup W_1$ and identifying each T_i with T_i'.

The embedding of D in W' is given by $D \cap W_0 = W(F, \bigcup_{i=1}^k \beta_i \cap N_r(x_0)) \subseteq W_0$ and $D \cap W_1 = W_1 \cap W(V, \{x_0'\}) \subseteq W_1$.

Finally, the manifold W is obtained from W' using Lemma 5.2. $\diamondsuit$

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