EMBEDDING MEDIAN ALGEBRAS IN MEDIAN METRIC SPACES

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Abstract. We show that any countable median algebra embeds in a median metric space. If the median algebra has finite rank, one can take the metric space to be compact and connected. A number of related embedding results are given.

1. Introduction

The notion of a median algebra was introduced in the 1950s and has found numerous applications since: see for example, [I, BaH, R, Bo2] and the references therein. They frequently arise as median metric spaces: examples of such are $\mathbb{R}$-trees and CAT(0) cube complexes (with the l^1 metric). Other examples arise from spaces of measured walls and asymptotic cones of various spaces.

Not every median algebra admits a metric in the usual sense. (For example, take any uncountable ordinal). However, we show that any countable median algebra admits a median metric. Moreover, if we generalise the notion of a metric so that it can take values in any ordered abelian groups, then any median alegebra admits a generalised median metric.

A median algebra has associated with it a "rank" in $\mathbb{N} \cup \{\infty\}$. Rank-1 median algebras can be thought of as treelike structures, much studied in their own right. (They have been variously known as "tree algebras", "Herrlich trees", "median pretrees" etc.) In this particular case, most of our main results are proven in [C], and many of our arguments are generalisations or variations on those presented there.

We will begin by giving some general definitions and relevant examples, and then proceed to state the main results. We refer to [I, BaH, R, Bo2] for elaboration on the general facts about median algebras which we will use.

Let M be a set equipped with a ternary operation, $\mu : M^3 \longrightarrow M$. We will assume throughout that μ is symmetric: that is to say, invariant under permutation of the arguments. We say that μ is a ***median*** if it satisfies $\mu(a, a, b) = a$ and $\mu(\mu(a, b, c), a, d) = \mu(\mu(a, b, d), a, c)$, for all $a, b, c, d \in M$. In this case, we refer to (M, μ) as a ***median algebra***. Various equivalent definitions can be found in [I, BaH, R, Bo2].

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Given $a, b, c \in M$, we say that c lies ***between*** a and b, and write $a.c.b$, to mean that $\mu(a,b,c) = c$. We write $[a,b] = [a,b]_M = \{x \in M \mid a.x.b\}$ for the median ***interval*** from a to b. One can verify that if $a, b, c \in M$, then $[a,b] \cap [b,c] \cap [c,a] = \{\mu(a,b,c)\}$.

A subset, $B \subseteq M$ is a ***subalgebra*** if it is closed under the median operation. We write $B \leq M$. Given any subset, $A \subseteq M$, we write $\langle A \rangle = \langle A \rangle_M$ for the subalgebra generated by A. It is easily seen (iterating the median operation) that if A is countable, then so is $\langle A \rangle$. Though it requires more work, one can also show that if A is finite, then so is $\langle A \rangle$.

A ***homomorphism*** between two median algebras is a map which respects the median operation. (Note that by an earlier observation, this is equivalent to saying that it preserves the betweenness relation.) It is a ***monomorphism*** (respectively ***isomorphism***) if it is injective (respectively bijective).

A two-point set, $\{0,1\}$, admits a unique median structure, with the median given by "majority vote" ($\mu(0,0,0) = \mu(0,0,1) = 0$ etc.). A direct product of median algebras is also a median algebra. In particular, $\{0,1\}^X$ is a median algebra for any set X. Note that we can identify $\{0,1\}^X$, with the power set, $\mathcal{P}(X)$, via characteristic functions, and the median is then given by $\mu(A, B.C) = (A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cup (B \cap C) \cup (C \cap A)$. If $\#X = n < \infty$, then we refer to $\{0,1\}^X \equiv \{0,1\}^n$ as an n-***cube***. The ***rank*** of a median algebra, M, is the maximal n such that M contains a (subalgebra isomorphic to an) n-cube. We set $\operatorname{rank}(M) = \infty$ if there is no upper bound.

An ***ordered abelian group*** is an abelian group, Λ, with a preferred subset, $P \subseteq \Lambda$, satisfying:

(O1): $0 \notin P$,

(O2): if $t, u \in P$, then $t + u \in P$, and

(O2): given any $t \in \Lambda$, either $t \in P$ or $-t \in P$.

We write $t < u$ to mean that $u - t \in P$. One sees that $<$ is a total order on Λ.

A Λ-***metric*** on a set M, is a map, $\rho = \rho_M : M^2 \longrightarrow \Lambda$ satisfying the usual axioms of a metric space: these make sense for values in any ordered abelian group. Thus an $\mathbb{R}$-metric is a metric in the usual sense. By default, a "metric space" will be an $\mathbb{R}$-metric space.

Given $a, b \in M$, write $[a,b]_\rho = \{x \in M \mid \rho(a,x) + \rho(x,b) = \rho(a,b)\}$. We say that ρ is ***median*** Λ-***metric*** if it satisfies:

(M): $(\forall a, b, c \in M)\, (\exists d \in M)\ [a,b]_\rho \cap [b,c]_\rho \cap [c,a]_\rho = \{d\}$.

In such a case, we set $\mu(a,b,c) = d$. A result of Sholander [S] tells us that (M, μ) is a median algebra with $[a,b]_\rho = [a,b]_M$ for all $a, b \in M$.

We will show:

Theorem 1.1. *Any median algebra, M, admits a structure as a Λ-median metric space for some ordered abelian group, Λ. If M is countable, we can take $\Lambda = \mathbb{R}$.*

Theorem 1.2. *A countable median algebra of finite rank admits a monomorphism to a compact connected median $\mathbb{R}$-metric space of the same rank.*

We also give another proof of the following result which can be found in [F]:

Theorem 1.3. *A median $\mathbb{R}$-metric space admits an isometric embedding into a complete connected median $\mathbb{R}$-metric space of the same rank.*

(Note that an isometric embedding is necessarily a median monomorphism.)

We make a few general observations.

One can show that the metric completion of a median metric space is again a median metric space of the same rank. A complete median metric space is geodesic. This means that any two points are connected by a geodesic; that is, an isometric embedding of a compact real interval.

A particular case is that of rank-1 median algebras. A connected rank-1 median metric space is an $\mathbb{R}$-***tree***. This last notion has numerous equivalent definitions. (For example, it is metric space in which any two points are connected by a unique arc, and this arc is isometric to a real interval.)

The rank-1 cases of Theorem 1.1 and 1.2 (modulo the compactness statement) are proven in [C].

2. Ultraproducts

The theory of ultraproducts will feature in some of our arguments (notably the first parts of Theorems 1.1 and 1.3). We recall some basic facts here.

Let $\mathcal{I}$ be a set, and $\mathcal{P}(I)$ its power set, ordered by inclusion. Given $U \in \mathcal{P}(I)$, write $U^* = I \setminus U$. An ***ultrafilter*** is a subset, $\mathcal{U} \subseteq \mathcal{P}(I)$ satisfying

(U1): $\varnothing \notin \mathcal{U}$,

(U2): for all $U, V \in \mathcal{U}$, $U \cap V \in \mathcal{U}$, and

(U3): if $U \in \mathcal{P}(I)$, then either $U \in \mathcal{U}$ or $U^* \in \mathcal{U}$.

If $P(i)$ is a predicate with free variable $i \in I$, we say that $P(i)$ holds for ***almost all*** i if $\{i \in I \mid P(i)\} \in \mathcal{U}$.

Let $\underline{X} := (X_i)_{i \in I}$ be a family of sets indexed by I. We will denote a typical element of $\prod \underline{X} := \prod_{i \in I} X_i$ as $\underline{x} = (x_i)_i$. We define an equivalence relation, $\sim$, on $\underline{X}$ by deeming $\underline{x} \sim \underline{y}$ if $x_i = y_i$ for almost all i. Write $x_\infty = [\underline{x}]$ for the equivalence class of $\underline{x}$, and write $X_\infty := \prod \underline{X}/\mathcal{U} := \prod \underline{X}/\sim = \{x_\infty \mid \underline{x} \in \prod \underline{X}\}$ for the quotient space. This is the ***ultraproduct*** of $\underline{X}$ with respect to $\mathcal{U}$.

This construction respects direct products. (We can naturally identify the ultraproduct of $(X_i \times Y_i)_i$ with $X_\infty \times Y_\infty$.) Viewing functions and relations as subsets of direct products, we can similarly pass to ultraproducts. Thus, if we have a family of functions, $f_i : X_i \longrightarrow Y_i$, we get a "limiting" function, $f_\infty : X_\infty \longrightarrow Y_\infty$.

If $x_\infty = X$ is fixed, we refer to $X_\infty = X^I/\mathcal{U}$ as an ***ultrapower***. There is a natural embedding of X into X_∞ (taking constant sequences). If X is finite, this is a bijection, and we can naturally identify X with X_∞.

Suppose we have a first order predicate, $P(x_\infty, y_\infty, z_\infty, \ldots)$, with arguments in X_∞. A general principle (due to Łoś) says that $P(x_\infty, y_\infty, z_\infty, \ldots)$ holds in x_∞ if and only if, the subsituted formula $P(x_i, y_i, z_i, \ldots)$ holds in X_i for almost all i. (Note that this is independent of the choice of representatives of $x_i, y_i, z_i, \ldots$.) This allows us to pass to ultraproducts of structures satsifying first order axioms. (Łoś's principle is easily verified directly in the cases where we apply it here.)

For example, suppose we have a family, (X_∞, ρ_i) of Λ-metric spaces. We can pass to an ultraproduct, (X_∞, ρ_∞), which is a Λ_∞-metric space. (Note that Λ_∞ is itself an ordered abelian group.) Moreover, if (almost) every (X_∞, ρ_∞) is median Λ-metric space, then (X_∞, ρ_∞) is a median Λ_∞-metric space of the same rank.

Ultafilters can arise in the following way.

Let $\mathcal{V} \subseteq \mathcal{P}(I)$ satisfy:

(F): Any finite intersection of elements of $\mathcal{V}$ is nonempty.

(This is taken to imply that $\varnothing \notin \mathcal{V}$.)

Lemma 2.1. *Suppose that $\mathcal{V} \subseteq \mathcal{P}(I)$ is maximal with respect to inclusion satisfying (F). Then $\mathcal{V}$ is an ultrafilter on I.*

Proof. We first verify (U3). Suppose, for contradiction, that $U \in \mathcal{P}(I)$ with $U, U^* \notin \mathcal{V}$. We claim that either $\mathcal{V} \cup \{U\}$ or $\mathcal{V} \cup \{U^*\}$ satisfies (F). For if not, we have $V_1, \ldots, V_m, V_1', \ldots V_n' \in \mathcal{V}$ with $U \cap V_1 \cap, \cdots, \cap V_m = U^* \cap V_1' \cap, \cdots, \cap V_n' = \varnothing$, giving the contradiction $V_1 \cap \cdots \cap V_m \cap V_1' \cap \cdots \cap V_n' = \varnothing$.

For (U2), suppose $U, V \in \mathcal{V}$. Then $U \cap V \cap (U \cap V)^* = \varnothing$, so by (F), $(U \cap V)^* \notin \mathcal{V}$, so by (U3), $U \cap V \in \mathcal{V}$. $\square$

Applyng Zorn's Lemma, we see that any family of subsets of I satisfying (F) lies in some ultrafilter on I.

Suppose now that $(I, \leq)$ is a directed set. (That is, $\leq$ is a partial order such that for all $x, y \in I$ there is some $z \in I$ with $z \leq x, y$.) Given $U \subseteq I$, write $\uparrow U$ for the set of all $z \in I$ such that $z \geq x$ for all $x \in U$. Thus $\uparrow (U \cup V) = (\uparrow U) \cap (\uparrow V)$. Moreover, if U is finite, then $\uparrow U \neq \varnothing$. Let $\mathcal{V}$ be the family of sets, $\uparrow U \subseteq \mathcal{P}(I)$, as U varies over all finite subsets of I. By the above, we see that $\mathcal{V}$ satisfies (F), and so $\mathcal{V} \subseteq \mathcal{U}$ for some ultrafilter, $\mathcal{U}$, on I.

Suppose X is any set, and let $(A_i)_{i \in I}$ be a family of non-empty finite subsets of X, indexed by some set I. We write $i \leq j$ to mean that $A_i \subseteq A_j$. We assume that $(A_i)_i$ cofinal in the set of all finite subsets of X. (That is, if $A \subseteq X$ is finite, there is some $i \in I$ with $A \subseteq A_i$.) In this case, $(I, \leq)$ is a directed set. It follows that there is an ultrafilter, $\mathcal{U}$, on I, such that $\uparrow U \in \mathcal{U}$ for all finite $U \subseteq I$. Let $A_\infty = \prod_i A_i/\mathcal{U}$. Note that if $A \subseteq X$ is finite, then $A \subseteq A_i$ for some i, and so in fact, $A \subseteq A_i$ for almost all i.

We can define a map, $\phi : X \longrightarrow A_\infty$ as follows. Given $x \in X$, set $x_i = x$ whenever $x \in A_i$ and set $x_i \in A_i$ arbitrarily otherwise. Note that the former case holds for almost all i and so $x_i = x$ for almost all i. Thus, x_∞ is well defined. We set $\phi(x) = x_\infty$.

We note that ϕ is injective. For suppose that $x \neq y$. Again, $x, y \in A_i$, and so $x_i \neq y_i$, for almost all i. Thus, $x_\infty \neq y_\infty$.

3. Proof of the first part of Theorem 1.1

We begin with some definitions.

Let M be a median algebra. A subset, $C \subseteq M$ is ***convex*** if $[a, b] \subseteq C$ for all $a, b \in C$. A ***wall*** of M, is an unordered partition, W, of M into two non-empty convex subsets. We will generally write $W = \{W^-, W^+\}$, though in general the choice of $+$ and $-$ is arbitrary. Given a subset, $A \subseteq M$, we write $A \pitchfork W$ to mean that $A \cap W^- \neq \varnothing$ and $W \cap W^+ \neq \varnothing$. We write $\mathcal{W}(M)$ for the set of all walls. We write $\mathcal{W}(M, A) = \{W \in \mathcal{W}(M) \mid A \pitchfork W\}$ and $\mathcal{W}^*(M, A) = \mathcal{W}(M) \setminus \mathcal{W}(M, A)$. We abbreviate $\mathcal{W}(M, a, b) := \mathcal{W}(M, \{a, b\})$. One can show that if $a \neq b$, the $\mathcal{W}(M, a, b) \neq \varnothing$.

Suppose that Π is a finite median algebra. By a ***width function*** we mean a map $\lambda : \mathcal{W}(\Pi) \longrightarrow (0, \infty) \subseteq \mathbb{R}$. We refer to $\lambda(W)$ as the ***width*** of W. Given any $\mathcal{W}' \subseteq \mathcal{W}(M)$, write $\lambda(\mathcal{W}') = \sum_{W \in \mathcal{W}'} \lambda(W)$. Given $a, b \in \Pi$ set $\rho_\lambda(a, b) = \lambda(\mathcal{W}(\Pi, a, b))$. One can show:

Lemma 3.1. *Let Π be a finite median algebra. Given any width funciton, λ, the map ρ_λ is a median metric on Π inducing the original median. Conversely, any median metric, ρ, on Π which induces the given median has the form $\rho = \rho_\lambda$ for a unique length function, $\lambda = \lambda_\rho$.*

In particular, taking λ to be identically 1, we get the ***combinatorial metric*** on Π. This is a $\mathbb{Z}$-metric.

Now let M be any median algebra. Let $(A_i)_{i \in I}$ be the family of finite subalgebras indexed by some set I. Writing $i \leq j$ to mean $A_i \subseteq A_j$, $(I, \leq)$ is a directed set, and $(A_i)_i$ is cofinal in the set of all finite subsets of M. Let $\mathcal{U}$ be an ultrafilter as constructed in Section 2. Let $A_\infty = \prod_i A_i/\mathcal{U}$, and let $\phi : M \longrightarrow A_\infty$ be the injective map constructed at the end of Section 2.

Now let ρ_i be the combinatorial metric on A_i with the median induced from M. Passing to ultraproducts, we get a $\mathbb{Z}_\infty$-metric, ρ_∞, on A_∞, where $\mathbb{Z}_\infty := \mathbb{Z}^I/\mathcal{U}$ is the ultrapower of $\mathbb{Z}$. Moreover, ρ_∞ is a median $\mathbb{Z}_\infty$-metric (the median property (M) being a first order property).

We claim that ϕ is a monomorphism. In fact, $x.y.z$ holds in M if and only if $\phi x.\phi y.\phi z$ holds in A_∞. To see this, let $\underline{x}, \underline{y}, \underline{z}$ be sequences defining $\phi x = x_\infty$, $\phi y = y_\infty$ and $\phi z = z_\infty$. For almost all i, we have $x_i, y_i, z_i \in A_i$, so that $x_i = x$, $y_i = y$ and $z_i = z$. Thus, $\phi x.\phi y.\phi z$, (i.e. $\rho_\infty(x_\infty, z_\infty) = \rho_\infty(x_\infty, y_\infty) + \rho_\infty(y_\infty, z_\infty)$) holds in A_∞ if and only if $x_i.y_i.z_i$ (i.e. $\rho_i(x_i, z_i) = \rho_i(x_i, y_i) + \rho_i(y_i, z_i)$) holds in

A_i for almost all i. The latter holds if and only if $x.y.z$ holds in A_i. Since, by assumption, ρ_i induces the orginal median, this is equivalent to $x.y.z$ in M, as claimed.

Finally, we set $N = \phi(M) \subseteq A_\infty$. This proves the first part of Theorem 1.1

4. Countable median algebras

We now prove the second part of Theorem 1.1. We first make some general observations about finite median algebras. Here all metrics will be $\mathbb{R}$-metrics.

Let Π be a finite median algebra, and $\Pi' \leq \Pi$ be a subalgebra. We have a map $\omega : \mathcal{W}(\Pi, \Pi') \longrightarrow \mathcal{W}(\Pi)$ defined by setting $\omega(W) = \{\Pi \cap W^-, \Pi \cap W^+\}$, where $W = \{W^-, W^+\}$.

Suppose that ρ is a median metric on Π. This restricts to a median metric, ρ', on Π'. By Lemma 3.1, we have $\rho = \rho_\lambda$ and $\rho' = \rho_{\lambda'}$, where λ, λ' are width functions on Π and Π' respectively. In fact, one can check that if $W \in \mathcal{W}(\Pi')$, then $\lambda'(W) = \lambda(\omega^{-1}W)$, where $\omega^{-1}W \subseteq \mathcal{W}(\Pi, \Pi')$ is the preimage of W under ω.

Conversely, given a median metric, $\rho' = \rho_{\lambda'}$ on Π', we can construct a median metric, $\rho = \rho_\lambda$ on Π extending ρ', by taking any width function λ on Π such that $\lambda(\omega^{-1}W) = \lambda'(W)$ for all $W \in \mathcal{W}(\Pi')$. (Note that we can define $\lambda|\mathcal{W}^*(\Pi, \Pi')$ arbitrarily.) We get:

Lemma 4.1. *If $\Pi' \leq \Pi$ are finite median algebras, then any median metric on Π' inducing the given median extends to a median metric on Π inducing the given median on Π.*

Now suppose that M is a countable median algebra. Write $M = \bigcup_{i=0}^{\infty} A_\infty$, where $A_0 \leq A_1 \leq A_2 \leq \cdots$ is an increasing union of finite subalgebras of M. By Lemma 4.1, we can inductively construct median metrics, ρ_i, on A_i, such that if $j \leq i$, then ρ_i restricts to ρ_j on $A_j \leq A_\infty$. If $x, y \in M$, then $x, y \in A_i$ for all sufficiently large i, and we set $\rho(x, y) = \rho_i(x, y)$. Since every finite subset of A lies in some A_i, we readily check that ρ is a median metric inducing the original median on M.

This completes the proof of Theorem 1.1.

5. Precompactness

In this section, we prove Theorem 2.1 modulo the connectedness statement.

First, we give some general definitions.

Given a finite median algebra, Π, we define the ***length*** of Π to be $L(\Pi) := \lambda(\mathcal{W}(\Pi))$, where λ is the associated width function, as given by Lemma 3.1.

If $\Pi' \leq \Pi$ is any subalgebra, then $L(\Pi') = \lambda'(\mathcal{W}(\Pi')) = \lambda(\mathcal{W}(\Pi, \Pi')) \leq \lambda(\mathcal{W}(\Pi))$. In fact, we see that $L(\Pi) = L(\Pi') + \lambda(\mathcal{W}^*(\Pi, \Pi'))$.

More generally, given any median algebra, M, we set $L(M) = \sup\{L(\Pi)\} \in [0,\infty]$, where Π varies over all finite subalgebras. (Note that by the earlier observation, this is consistent with the original definition if M is finite.) Clearly, if $M \leq N$, then $L(M) \leq L(N)$.

Lemma 5.1. *Suppose that M is a median metric space, and $M \leq N$ is a dense subalgebra (in the topological sense). Then $L(M) = L(N)$.*

Proof. We certainly have $L(M) \leq L(N)$. We therefore want so show that $L(A) \leq L(M)$ for any finite subalgebra, $A \leq N$.

Let $n = \#A$. Given any $\epsilon > 0$, there is a map $f : A \longrightarrow M$ with $\rho(a, fa) \leq \epsilon/n$ for all $a \in A$. Let $B = \langle f(A)\rangle_M$ and let $C = \langle A \cup B\rangle_N$. Thus, $B \leq M$ and $C \leq N$ are finite subalgebras. In particular, $L(B) \leq L(M)$. Moreover, $L(C) = L(B) + \lambda(\mathcal{W}^*(C, B))$, where λ is the width function on C given by the median metric, ρ, restricted to C.

If $W \in \mathcal{W}^*(C, B)$, then (up to swapping $+, -$) we have $f(A) \subseteq B \subseteq W^-$. Also, $A \cap W^+ \neq \varnothing$ (otherwise, $A \cup B \subseteq W^-$ giving the contradiction that $C \subseteq W^-$). If $a \in A \cap W^+$, then $W \in \mathcal{W}(C, a, fa)$. In other words, we have $\mathcal{W}^*(C, B) \subseteq \bigcup_{a\in A} \mathcal{W}(C, a, fa)$. Thus, $\lambda(\mathcal{W}^*(C, B)) \leq \sum_{a\in A} \lambda(\mathcal{W}(C, a, b)) = \sum_{a\in A} \rho(a, fa) \leq n(\epsilon/n) = \epsilon$. Thus, $L(A) \leq L(C) \leq L(B) + \epsilon \leq L(M) + \epsilon$. Letting $\epsilon \to 0$, we get $L(A) \leq L(M)$ as required. $\square$

Recall that a subset, $A \subseteq M$, of a metric space, M, is ϵ-***separated*** if $\rho(x, y) \geq \epsilon$ for all distinct $x, y \in A$. We say that M is ***precompact*** (or ***totally bounded***) if, for all $\epsilon > 0$, every ϵ-separated subset is finite. We claim:

Lemma 5.2. *Let M be a median metric space with* $\operatorname{rank}(M) = \nu < \infty$ *and $L(M) < \infty$. Then M is precompact.*

In fact, we will show that if $A \subseteq M$ is ϵ-separated, then $\#A$ is bounded by an explicit function of ϵ, ν and $L(M)$. Note that, replacing M by $\langle A\rangle_M$, it is enough to prove Lemma 5.2 when M is finite. We will use the following observation.

Lemma 5.3. *Let Π be a finite median metric space of rank at most $\nu < \infty$, and diameter at least $D > 0$. Then there is a wall, $W \in \mathcal{W}(\Pi)$ with $L(W^-) \leq L(\Pi) - D/2\nu$ and $L(W^+) \leq L(\Pi) - D/2\nu$.*

Proof. Let $a, b \in \Pi$ with $\rho(a, b) \geq D$. Given $W \in \mathcal{W}(\Pi, a, b)$, we write $W = \{W^-, W^+\}$ with $a \in W^-$ and $b \in W^+$. Given $W, W' \in \mathcal{W}(\Pi, a, b)$, write $W \leq W'$ to mean that $W^- \subseteq (W')^-$ (or equivalently, $(W')^+ \subseteq W^+$). Thus, $\leq$ is a partial order on $\mathcal{W}(\Pi, a, b)$. Now any antichain in $\mathcal{W}(\Pi, a, b)$ has cardinality at most ν. (This is because any antichain is a set of pairwise crossing walls, and the cardinality of any such set in any median algebra is bounded by the rank: see for example [Bo2].) Now Dilworth's Lemma [D] tells that we can write $\mathcal{W}(\Pi, a, b)$ as a disjoint union, $\mathcal{W}(\Pi, a, b) = \bigsqcup_{i=1}^{\nu} \mathcal{W}_i$, of ν chains, $\mathcal{W}_1, \ldots, \mathcal{W}_\nu$. Now $\sum_{i=1}^{\nu} \lambda(\mathcal{W}_i) = \lambda(\mathcal{W}(\Pi, a, b)) = \rho(a, b) \geq D$. Thus, there is some i with

$\lambda(\mathcal{W}_i) \geq D/\nu$. Write the elements of $\mathcal{W}_i$ as $W_1 < W_2 < \cdots < W_m$. Now, there is some j such that $\lambda(\mathcal{W}^-), \lambda(\mathcal{W}^+) \geq D/2\nu$, where $\mathcal{W}^- = \{W_j \mid j \leq m\}$ and $\mathcal{W}^+ = \{W_j \mid j \geq m\}$. Now, $\mathcal{W}(\Pi, W_j^-) \subseteq \mathcal{W}(\Pi) \setminus \mathcal{W}^+$, and so $L(W_j^-) = \lambda(\mathcal{W}(\Pi, W_j^-)) \leq \lambda(\mathcal{W}(\Pi)) - \lambda(\mathcal{W}^+) \leq L(\Pi) - D/2\nu$. Similarly, $L(W_j^+) = L(\Pi) - D/2\nu$. We set $W = W_j$. □

Lemma 5.4. *Let $k \in \mathbb{N}$ and $\epsilon > 0$. Let Π be a finite median metric space of rank $\nu < \infty$, and with $L(\Pi) \leq k\epsilon/2\nu$. If $A \subseteq \Pi$ is ϵ-separated, then $\#A \leq 2^k$.*

Proof. We prove this by induction on k. We can assume that $\#A \geq 2$, so that the diameter of Π is at least ϵ. By Lemma 5.3, there is some $W \in \mathcal{W}(\Pi)$ with $L(W^{\pm}) \leq L(\Pi) - \epsilon/2\nu \leq (k-1)\epsilon/2\nu$. By the inductive hypothesis, we have $\#(A \cap W^{\pm}) \leq 2^{k-1}$, so $\#A \leq 2^k$ as required. □

Proof of Lemma 5.2. We can suppose that M is finite. Given $\epsilon > 0$, let $k = \lfloor 2\nu L(M)/\epsilon \rfloor + 1$. By Lemma 5.4, if $A \subseteq M$ is ϵ-separated, then $\#A \leq 2^k$. □

We noted earlier that the completion of any median metric space is a median metric space of the same rank. Moreover, a complete precompact metric space is compact. Therefore, putting to together Lemmas 5.1 and 5.2, we get:

Lemma 5.5. *Let M be a median metric space of finite rank and finite length. Then the completion, $\bar{M}$, is compact.*

Now let M be a countable median algebra. We write M as a countable union of finite subalgebras, $M = \bigcup_{i=0}^{\infty} A_{\infty}$, as in Section 4. Now when extending the metric ρ_i to ρ_{i+1}, we can set the width function $\lambda = \lambda_{\rho_{i+1}}$ arbitrarily on $\mathcal{W}^*(A_{i+1}, A_i)$. In particular, we can assume that $\lambda(\mathcal{W}^*(A_{i+1}, A_i)) \leq 2^{-(i+1)}$. Thus, $L(A_{i+1}) \leq L(A_i) + 2^{-(i+1)}$. We can also take $L(A_0) \leq 1$. Thus, $L(A_i) \leq 2$ for all i. Now any finite subalgebra, $A \leq M$, lies in A_i for all sufficiently large i, so $L(A) \leq L(A_i) \leq 2$, and so $L(M) \leq 2$. Lemma 5.5 now tells us that the completion, $\bar{M}$, is compact.

We have shown that any countable median algebra embeds into a compact median metric space of the same rank and inducing the original median. This proves Theorem 1.2 modulo the connectedness statement.

6. Connectedness

In this section, we complete the proof of Theorem 1.2.

Let Π be a finite median metric space. Given $W \in \mathcal{W}(\Pi)$, let $J(W) = [0, \lambda(W)] \subseteq \mathbb{R}$, and let $\Psi = \Psi(\Pi) = \prod_{W \in \mathcal{W}(\Pi)} J(W) \subseteq \mathbb{R}^{\#\mathcal{W}(\Pi)}$. We equip Ψ with the l^1 metric, so that it is a median metric space. There is a natural embedding of Π into Ψ. (If $a \in \Pi$, we take the $J(W)$-coordinate of a to be 0 if $a \in W^-$ and to be $\lambda(W)$ if $a \in W^+$.) One checks that this is an isometric embedding hence a monomorphism. Identifying Π with its image, we can view it as a subalgebra $\Pi \leq \Psi$.

Let $\mathcal{F}(\Psi)$ be the set of faces of Ψ (viewed as a convex polyhedron in $\mathbb{R}^{\#\mathcal{W}(\Pi)}$). Let $\mathcal{F}(\Pi) \subseteq \mathcal{F}(\Psi)$ be the set of faces all of whose vertices lie in Π, and let $\Delta = \Delta(\Pi) = \bigcup \mathcal{F}(\Pi) \subseteq \Psi$. Thus, Δ is a subcomplex of Ψ. In fact, one can verify that Δ is connected, and a subalgebra of Ψ. (This is discussed in [Bo2] for example.) In the induced l^1 metric, Δ is therefore a compact connected median metric space, with $\Pi \subseteq \Delta$, isometrically embedded as a subalgebra. We refer to Δ as the ***realisation*** of Π.

Suppose $A \leq \Delta$ is a finite subalgebra. Let $B = \langle A \cup \Pi \rangle_\Delta$. This is also a finite subalgebra. Moreover, $\mathcal{W}(B, \Pi) = \mathcal{W}(B)$, and so $L(A) \leq L(B) = L(\Pi)$. It follows that $L(\Delta) = L(\Pi)$. (In fact, all we have really used here is that Δ is the smallest convex subset of Δ containing Π.)

One can also show that if M is any complete connected median metric space, and $\Pi \leq M$ is any finite subalgebra, then the inclusion of Π into M extends to an isometric embedding of $\Delta(\Pi)$ into M. A proof can be found in [Bo2]. Such a map is necessarily a median monomorphism. (Of course, it is not in general unique.)

In summary, we get:

Lemma 6.1. *Let Π be a finite median metric space. Then Π embeds into a compact connected median metric space, $\Delta(\Pi)$, with $L(\Delta(\Pi)) = L(\Pi)$. Moreover, if $\Pi' \leq \Delta(\Pi)$ is any finite subalgebra, then the inclusion of Π' into $\Delta(\Pi)$ extends to an isometric embedding of $\Delta(\Pi')$ into $\Delta(\Pi)$.*

Next, suppose that M is a countable median algebra. We revisit the argument given in Sections 4 and 5. Recall that we have put a median metric on M so that $L(M) \leq 2$, and that we have represented M as an increasing union of finite subalgebras, $A_0 \leq A_1 \leq A_2 \leq \cdots$. Now let $\Delta_i = \Delta(A_i)$. By Lemma 6.1, we can isometrically embed Δ_i into Δ_{i+1} for all i. In this way, if $j \leq i$, then $\Delta_j \leq \Delta_i$. Let $\Delta = \bigcup_{i=0}^{\infty} \Delta_i$. This is a median metric space. Note that if $A \leq \Delta$ is any finite subalgebra, then $A \leq \Delta_i$ for all sufficiently large i. Thus, $L(A) \leq L(\Delta_i) \leq L(M) \leq 2$. Now let N be the metric completion of Δ. This is connected. Moreover, by Lemma 5.5, N is compact.

This completes the proof of Theorem 1.2.

7. Proof of Theorem 1.3

We give another proof of the result of [F], namely Theorem 1.3. We begin with some further observations about ultraproducts.

Let I be a set, and $\mathcal{U}$ and ultrafilter on I. Let $[0, \infty)_\infty := [0, \infty)^I/\mathcal{U}$ be the ultrapower of the ray, $[0, \infty) \subseteq \mathbb{R}$. We can identify $[0, \infty)_\infty$ with the set of non-negative elements in the ordered abelian group, $\mathbb{R}_\infty$. We can also identify $[0, \infty)$ as a subset of $[0, \infty)_\infty$. Let $[0, \infty]$ be the one-point compactification of $[0, \infty)$. Setting $t + \infty = \infty$ for all $t \in [0, \infty]$, this is a semigroup. The identity map on $[0, \infty)$ extends to a semigroup homomorphism, $\pi : [0, \infty)_\infty \longrightarrow [0, \infty]$. This can be described as follows. Let $\underline{t} = (t_i)_i \in [0, \infty)^I$. From the compactness of $[0, \infty]$, one can show that there is a unique $l \in [0, \infty]$ such that $t_i \to l$. If $l \neq \infty$, this means that for all $\epsilon > 0$, we have $|t_i - l| \leq \epsilon$ for almost all i. If $l = \infty$, this means that for all $k \in [0, \infty)$, we have $t_i \geq k$ for almost all i. We write $l = \lim \underline{t}$. Note that if $\underline{t} \sim \underline{u}$, then $\lim \underline{t} = \lim \underline{u}$. If $t_\infty = [\underline{t}] \in [0, \infty)_\infty$, we can therefore set $\pi(t_\infty) = \lim \underline{t}$. Note that if $t \in [0, \infty)$, then $t = \lim \underline{t}$, where $\underline{t}$ is the constant sequence at t. Therefore $\pi|[0, \infty)$ is the identity as claimed.

Suppose now that (X_i, ρ_i) is a family of $\mathbb{R}$-metric spaces, indexed by $i \in I$. Let (X_∞, ρ_∞) be the ultraproduct. This is an $\mathbb{R}_\infty$-metric space, with metric $\rho_\infty : X_\infty^2 \longrightarrow [0, \infty)_\infty$. We can postcompose this with the map π to give us a "distance" function, $\pi\rho_\infty : X_\infty^2 \longrightarrow [0, \infty]$.

We can define an equivalence relation, $\simeq$, on X_∞, by writing $x \simeq y$ to mean $\pi\rho_\infty(x, y) < \infty$. We refer to a $\simeq$-class as a ***component*** of X_∞. Let Y be one such component. We thus have a map $\pi\rho_\infty : Y^2 \longrightarrow [0, \infty)$. One checks that this is a pseudometric on Y.

We now define another equivalence relatation, $\approx$, on Y by setting $x \approx y$ to mean that $\pi\rho_\infty(x, y) = 0$. Let $\hat{Y} := Y/\!\approx$ be the quotient space. Given $x \in Y$, we write $\hat{x} \in \hat{Y}$ for the class of x. We have a well defined map, $\hat{\rho} : \hat{Y}^2 \longrightarrow [0, \infty)$, where $\hat{\rho}(\hat{x}, \hat{y}) = \pi\rho_i(x, y)$, for all $x, y \in Y$. One checks that $(\hat{Y}, \hat{\rho})$ is a metric space in the usual sense. This is the ***hausdorffification*** of $(Y, \pi\rho_\infty)$.

Given $a \in \hat{Y}$, we can choose a representative, a_∞, of a in Y (that is, $a = \hat{a}_\infty$), and then choose a representative, $\underline{a} = (a_i)_i$ of a_∞ in $\prod_i X_i$. We write $a_i \to a$ to mean this. From the construction, we see that if $a_i \to a$ and $b_i \to b$, then $\rho_i(a_i, b_i) \to \hat{\rho}(a, b)$ in $[0, \infty)$ in the sense of ultrafilters defined earlier. That is to say, given any $\epsilon > 0$ we have $|\rho_i(a_i, b_i) - \hat{\rho}(a, b)| \leq \epsilon$ for almost all i.

We claim:

Lemma 7.1. *Suppose that (X_i, ρ_i) is a median metric space for (almost) all i. Let Y be a component of X_∞, and let $\hat{Y}$ be its hausdorffification. Then $(\hat{X}, \hat{\rho})$ is a median metric space.*

For this will need the following general fact about median metrics.

Given a metric space, (M, ρ), elements, $a, b \in M$, and $\epsilon \geq 0$, write $[a, b]_\rho^\epsilon = \{x \in M \mid \rho(a, x) + \rho(x, b) \leq \rho(a, b) + \epsilon\}$. (Thus $[a, b]_\rho^0 = [a, b]_\rho$ as defined earlier.)

A proof of the following can be found in [Bo1].

Lemma 7.2. *Let (M, ρ) be a median metric space, let $a, b, c \in M$, and $d, e \in [a, b]_\rho^\epsilon \cap [b, c]_\rho^\epsilon \cap [c, a]_\rho^\epsilon$ for some $\epsilon \geq 0$. Then $\rho(d, e) \leq 2\epsilon$.*

Proof of Lemma 7.1. Let $a, b, c \in \hat{Y}$. We want to show that $[a, b]_{\hat{\rho}} \cap [b, c]_{\hat{\rho}} \cap [c, a]_{\hat{\rho}}$ consists of a single point of $\hat{Y}$.

We choose $(a_i)_i$, $(b_i)_i$ and $(c_i)_i$ with $a_i \to a$, $b_i \to b$ and $c_i \to c$, in the sense defined earlier. Let $d_i = \mu_i(a_i, b_i, c_i)$, where μ_i is the median on X_i. Then for all i, we have $\rho_i(a_i, d_i) + \rho_i(d_i, b_i) = \rho_i(a_i, b_i)$, so passing to limits, we get $\hat{\rho}(a, d) = \hat{\rho}(d, b) = \hat{\rho}(a, b)$. (Note that $\rho_\infty(a_\infty, d_\infty) \leq \rho_\infty(a_\infty, b_\infty) < \infty$ so $d_\infty \in Y$.) In other words, we have $d \in [a,b]_{\hat{\rho}}$ in $\hat{Y}$. Similarly, $d \in [b,c]_{\hat{\rho}}$ and $d \in [c,a]_{\hat{\rho}}$. This proves existence of medians in $\hat{Y}$.

For uniqueness, suppose $e \in [a,b]_{\hat{\rho}} \cap [b,c]_{\hat{\rho}} \cap [c,a]_{\hat{\rho}}$. Let $e_i \to e$. Now $\hat{\rho}(a,e) + \hat{\rho}(e,b) = \hat{\rho}(a,b)$, so $\rho_i(a_i,e_i) + \rho_i(e_i,b_i) - \rho_i(a_i,b_i) \to 0$. Thus, given $\epsilon > 0$, we have $\rho_i(a_i,e_i) + \rho_i(e_i,b_i) \leq \rho_i(a_i,b_i) + \epsilon$ for almost all i. That is, $e_i \in [a_i,b_i]^{\epsilon}_{\rho_i}$. Similarly, $e_i \in [b_i,c_i]^{\epsilon}_{\rho_i}$ and $e_i \in [c_i,a_i]^{\epsilon}_{\rho_i}$ for almost all i. By Lemma 7.2, this implies $\rho_i(d_i,e_i) \leq 2\epsilon$. In other words, $\rho_i(d_i,e_i) \longrightarrow 0$, so $\hat{\rho}(d,e) = 0$, so $d = e$ as required. $\square$

We also note:

Lemma 7.3. *If (X_i,ρ_i) is a geodesic metric space for (almost) all i, then $(\hat{Y},\hat{\rho})$ is a geodesic meric space.*

Proof. Let $a,b \in \hat{Y}$, and choose $(a_i)_i$, $(b_i)_i$ with $a_i \to a$ and $b_i \to b$. Ler $r = \hat{\rho}(a,b)$ and let $r_i = \rho_i(a_i,b_i)$. Thus $r_i \to r$. For any i, we have an ρ_i-lipschitz path, $\alpha_i : [0,1] \longrightarrow X_i$, from a_i to b_i. Given $t \in [0,1]$, define $\alpha(t)$ by $\alpha_i(t) \to \alpha(t)$. This gives a map $\alpha : [0,1] \longrightarrow \hat{Y}$, and one checks that this is an r-lipschitz path from a to b, that is, a geodesic. $\square$

We can finally prove Theorem 1.3.

Let (M,ρ) be any median metric space. Let $(A_i)_{i\in I}$ be the family of finite subalgebras of M, and let $\mathcal{U}$ be the ultrafilter on I, as discussed in Sections 2 and 3. This time, we take the metric, ρ_i, on M induced by ρ. Let $\Delta_i = \Delta(A_i)$ as constructed in Section 5, and let Δ_∞ be the ultralimit. This is a median $\mathbb{R}$-metric space. As in Section 3, we have a injective map $\phi : M \longrightarrow A_\infty$, which is a median monomorphism. Now $\phi(M)$ lies in some component, Y, of $(\Delta_\infty, \pi\rho_i)$. Let $(\hat{Y},\hat{\rho})$ be the hausdorffification of Y. By Lemmas 7.1 and 7.3, $\hat{Y}$ is a connected median metric space. Postcomposing ϕ with the quotient map $Y \longrightarrow \hat{Y}$, we get a map $\psi : M \longrightarrow \hat{Y}$. By construction, if $x,y \in M$, then $x,y \in A_i$ for almost all i. We have $x_i = x$ and $y_i = y$ for such i, and so $\rho_i(x_i,y_i) = \rho(x,y)$. Therefore, $\hat{\rho}(\psi x, \psi y) = \rho_\infty(x_\infty, y_\infty) = \rho(x,y)$. In other words, ψ is an isometric embedding.

We can now set N to be the metric completion of $\hat{Y}$. (In many cases, for example if I is countable, it turns out that $\hat{Y}$ is already complete.) Thus, N is a complete connected median metric space as required.

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