# RIGIDITY OF THE STRONGLY SEPARATING CURVE GRAPH

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Abstract. We define the strongly separating curve graph to be the full subgraph of the curve graph of a compact orientable surface, where the vertex set consists of all separating curves which do not bound a three-holed sphere. We show that, for all but finitely many surfaces, any automorphism of the strongly separating curve graph is induced by an element of the mapping class group.

## 1. Introduction

The main aim of this paper is to prove a rigidity result (Theorem 1.1) for certain curve graphs associated to compact orientable surfaces. It is a variation on some well known results in this direction. Our main motivation for this particular statement is its application to the quasi-isometric rigidity of the Weil-Petersson metric.

Let $\Sigma$ be a compact orientable surface. We write $g(\Sigma)$ for its genus, and $p(\Sigma)$ for the number of boundary components. The *complexity*, $\xi(\Sigma)$, of $\Sigma$ is defined by $\xi(\Sigma) = 3g(\Sigma)+p(\Sigma)-3$. (It equals the number of disjoint simple closed curves needed to cut $\Sigma$ into a collection of 3-holed spheres.)

Let $\mathcal{G}(\Sigma)$ be the curve graph associated to $\Sigma$; that is, the 1-skeleton of the curve complex as defined in [H]. It has vertex set $C(\Sigma)$, the set of non-trivial non-peripheral simple closed curves in $\Sigma$, defined up to homotopy. Two elements of $C(\Sigma)$ are deemed adjacent if they can be homotoped to be disjoint. Note that the mapping class group, $\mathrm{Map}(\Sigma)$, acts cofinitely on $\mathcal{G}(\Sigma)$. The rigidity theorems of [Iv, Ko, L] tell us (in particular) that if $\xi(\Sigma) \geq 2$, then any automorphism of $\mathcal{G}(\Sigma)$ is induced by an element of $\mathrm{Map}(\Sigma)$. (Note that, since the curve complex is a flag complex, this is equivalent to the same statement for the curve complex.)

There are a number of variations of this. Given a subset, $A \subseteq C(\Sigma)$, we write $\mathcal{G}(\Sigma, A)$ for the full subgraph of $\mathcal{G}(\Sigma)$ with vertex set $A$. If $A$ is $\mathrm{Map}(\Sigma)$-invariant, then $\mathrm{Map}(\Sigma)$ also acts on $\mathcal{G}(\Sigma, A)$. We say that

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$\mathcal{G}(\Sigma, A)$ is *rigid* if every automorphism is induced by an element of $\mathrm{Map}(\Sigma)$.

For example, if $C_s(\Sigma)$ is the set of separating curves, we refer to $\mathcal{G}_s(\Sigma) = \mathcal{G}(\Sigma, C_s(\Sigma))$ as the *separating curve graph*. (Note that if $g = 0$, then this is the same as $\mathcal{G}(\Sigma)$.) The results of [BrM, Ki], together with that cited above for planar surfaces, tell us $\mathcal{G}_s(\Sigma)$ is rigid if $g(\Sigma) \geq 3$ or ($g(\Sigma) = 2$ and $p(\Sigma) \geq 2$) or ($g(\Sigma) = 1$ and $p(\Sigma) \geq 2$) or ($g(\Sigma) = 0$ and $p(\Sigma) \geq 5$).

We remark that the non-separating curve graphs, $\mathcal{G}(\Sigma, C(\Sigma) \backslash C_s(\Sigma))$, of a large class of surfaces of genus at least 2 are also rigid [Ir], though this is not directly relevant to the present paper.

Let $C_0(\Sigma) \subseteq C_s(\Sigma)$ be the set of curves which bound some three-holed sphere in $\Sigma$. Let $C_{ss}(\Sigma) = C_s(\Sigma) \setminus C_0(\Sigma)$, and let $\mathcal{G}_{ss}(\Sigma) = \mathcal{G}(\Sigma, C_{ss}(\Sigma))$. We refer to elements of $C_{ss}(\Sigma)$ as *strongly separating* curves and to $\mathcal{G}_{ss}(\Sigma)$ as the *strongly separating curve graph*.

We will show here that $\mathcal{G}_{ss}(\Sigma)$ is rigid in all but finitely many cases:

**Theorem 1.1.** *If $g(\Sigma) + p(\Sigma) \geq 7$, then $\mathcal{G}_{ss}(\Sigma)$ is rigid.*

Note that if $p(\Sigma) \leq 1$, then $\mathcal{G}_{ss}(\Sigma) = \mathcal{G}_s(\Sigma)$, and so this is covered by the results of [BrM, Ki].

This still leaves unresolved about a dozen cases, which I suspect are also rigid. One can probably deal with a few more cases with some elaboration on the arguments here, though a complete answer may require new ideas.

It is natural to ask more generally for what classes of subsets $A \subseteq C_s(\Sigma)$ is $\mathcal{G}(\Sigma, A)$ rigid. (Note there are only finitely many possibilities for $A$ for any given topological type.)

The motivation for studying this particular case is the application given in [Bo2] to the Weil-Petersson metric on Teichmüller space. There it was shown that the rigidity of $\mathcal{G}_{ss}(\Sigma)$ implies the quasi-isometric rigidity of the Weil-Petersson metric associated to $\Sigma$. In view of Theorem 1.1, this holds whenever $g(\Sigma) + p(\Sigma) \geq 7$. In particular, Theorem 1.1, together with the results of that paper, shows that in all but at most finitely many cases, the Weil-Petersson space is quasi-isometrically rigid.

We remark that the quasi-isometric rigidity of the Teichmüller metric on Teichmüller space has been proven independently in [EMR] and [Bo1] (by different methods). For this, one makes use of the rigidity of the curve graphs [Iv, Ko, L] (in place of the strongly separating curve graphs as needed for the Weil-Petersson metric).

In the course of proving the main result of this paper, we also show that most strongly separating curve graphs are distinct (see Proposition 5.2).

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## 2. Outline of proof

We begin by introducing some terminology and notation used throughout the paper.

Let $\Sigma$ be a compact surface. We assume that $\xi(\Sigma) \geq 2$. Given a curve $\gamma \in C_s(\Sigma)$ we will (by slight abuse of terminology) use the term *complementary component* to refer to the closure of a connected component of $\Sigma \setminus \gamma$. We write $B(\gamma)$ for the complementary component which has smaller complexity. (We will only use this notation when it is unambiguous.) We write $C_i(\Sigma) \subseteq C_s(\Sigma)$ for the set of $\gamma \in C_s(\Sigma)$ for which $B(\gamma)$ has complexity at most $i$. When $i = 0$, $B(\gamma)$ is an $S_{0,3}$. (Recall that $C_0(\Sigma)$ was defined in this way in Section 1.) Given distinct $\alpha, \beta \in C_s(\Sigma)$, we write $\alpha < \beta$ to mean that $B(\alpha) \subseteq B(\beta)$. We will always assume curves in $C_s(\Sigma)$ to be realised in general position, and with minimal intersection. Given $\alpha, \beta \in C_{ss}(\Sigma)$, we write $\iota(\alpha, \beta) = |\alpha \cap \beta|$ for the geometric intersection number. The following notion will be central to the proof:

**Definition.** (When $p(\Sigma) \geq 5$.) We say that $\alpha, \beta \in C_{ss}(\Sigma)$ form a *surrounding pair* if $B(\alpha)$ and $B(\beta)$ are both $S_{0,4}$'s and $B(\alpha) \cap B(\beta)$ is an $S_{0,3}$.

(We will need to modify this definition slightly when $p(\Sigma) \leq 4$, as we discuss in Section 6.)

Note that $\partial(B(\alpha) \cap B(\beta))$ gives a curve $\omega \in C_0(\Sigma)$ satisfying $\omega < \alpha$ and $\omega < \beta$. Indeed, $\omega$ is uniquely determined by this property, and we say that the pair $\alpha, \beta$ *surrounds* $\omega$.

**Definition.** We say that $\alpha, \beta, \gamma \in C_{ss}(\Sigma)$ form a *surrounding triple* if any pair of them form a surrounding pair, and there is some (necessarily unique) $\omega \in C_0(\Sigma)$, with $\omega < \alpha$, $\omega < \beta$ and $\omega < \gamma$.

(Modulo the definition of "surrounding pair", the definition of "surrounding triple" will remain unchanged when $p(\Sigma) \leq 4$.)

We will say that we can *recognise* a given property of a collection of curves in $C_{ss}(\Sigma)$ if it is preserved under any automorphism of $\mathcal{G}_{ss}(\Sigma)$. Thus, by the definition of $\mathcal{G}_{ss}(\Sigma)$, we can recognise disjointness of

curves. The ultimate goal will be to show that we can recognise all combinatorial (i.e. $\mathrm{Map}(\Sigma)$-invariant) properties.

In particular, an intermediate goal is to recognise surrounding pairs and surrounding triples. This will allow us to "reconstruct" the graph $\mathcal{G}_s(\Sigma)$ from $\mathcal{G}_{ss}(\Sigma)$ in a canonical way. (We have already observed that a surrounding pair determines an element of $C_0(\Sigma)$, which is a step in that direction.) Thus, any automorphism of $\mathcal{G}_{ss}(\Sigma)$ extends to an automorphism to $\mathcal{G}_s(\Sigma)$, and we can apply the results of [Ko, BrM, Ki] to see that it is induced by an element of $\mathrm{Map}(\Sigma)$.

In Sections 3 and 4, we begin by considering the case where $\Sigma$ is an $S_{0,7}$. The key point here is that there is only one heptagon (that is 7-cycle) in $\mathcal{G}_{ss}(S_{0,7})$ up to the action of $\mathrm{Map}(S_{0,7})$ (see Proposition 3.1). A surrounding pair can now be recognised as a pair of vertices at distance 2 apart in some heptagon. In Section 4, we proceed to show that $\mathcal{G}_{ss}(S_{0,7})$ is rigid.

For the general case, we need to recognise the topological type of a multicurve in the surface $\Sigma$. This argument is largely independent of the $S_{0,7}$ case, and is discussed in Section 5. In Section 6 we then combine this with what we know about $S_{0,7}$ to prove Theorem 1.1 in general (for $g(\Sigma) + p(\Sigma) \geq 7$) — except, that is, when $\Sigma$ is an $S_{0,8}$, which we will treat as a special case in Section 7.

## 3. Heptagons in the 7-holed sphere

We begin with a description of 7-cycles (or "heptagons") in the separating curve graph of $S_{0,7}$. In general, by an *$n$-cycle* in $\mathcal{G}_{ss}(\Sigma)$, we mean a cyclically ordered sequence of $n$ vertices, where consecutive vertices are adjacent. We refer to it as an *odd* or *even* cycle depending on whether $n$ is odd or even. Note that any shortest odd cycle is necessarily isometrically embedded.

For the purposes of this and the next section, it will be convenient to view $S_{0,7}$ as (the complement of) the 2-sphere, $S$, with a set of 7 preferred points, $\Pi \subseteq S$. (In other words, $S$ is obtained by collapsing each boundary component of $S_{0,7}$ to a point of $\Pi$. We can recover $S_{0,7}$ by removing a small open disc about each point of $\Pi$.)

Note that if $\gamma \in C_{ss}$, then $\gamma$ bounds a disc, $B(\gamma)$, with $|B(\gamma) \cap \Pi| = 3$. We write $\pi(\gamma) = B(\gamma) \cap \Pi$. Note that, $\alpha, \beta \in C_{ss}$ are adjacent if and only if we can realise $\alpha, \beta$ so that $B(\alpha) \cap B(\beta) = \emptyset$. We say that two curves are *$n$-distant* if they are a distance exactly $n$ apart in $\mathcal{G}_{ss}$.

Note that if $\iota(\alpha, \beta) = 2$, then $\alpha \cup \beta$ cuts $S$ into four discs. In particular, $B(\alpha) \cap B(\beta)$, $B(\alpha) \cup B(\beta)$, $B(\alpha) \setminus B(\beta)$ and $B(\beta) \setminus B(\alpha)$ are all discs.

It is easily seen that $\mathcal{G}_{ss}(S_{0,7})$ has no 3-cycles. Also it has no 4-cycles. (For if $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ were a 4-cycle, $B(\gamma_1) \cup B(\gamma_3)$ and $B(\gamma_2) \cup B(\gamma_4)$ would be disjoint connected subsurfaces of $S$, each containing at least four points of $\Pi$, which clearly is not possible.) We will also see (Lemma 3.3) that $\mathcal{G}_{ss}(S_{0,7})$ has no 5-cycles. This implies that any 7-cycle must be isometrically embedded.

In fact, $\mathcal{G}_{ss}(S_{0,7})$ does contain 6-cycles. (For example, take disjoint discs, $D_1, D_2, F_1, F_2, F_3$, in $S$ so that each $|D_i \cap \Pi| = 2$ and each $|F_j \cap \Pi| = 1$. Now connect each $D_i$ to each $F_j$ by a set of six disjoint arcs, $a_{ij}$. Let $\beta_{ij}$ be the boundary of a regular neighbourhood of $D_i \cup a_{ij} \cup F_j$. Then $\beta_{11}, \beta_{23}, \beta_{12}, \beta_{21}, \beta_{13}, \beta_{22}$ is a 6-cycle.) We will however focus on the 7-cycles, since these are more symmetrical, and will serve for our purposes.

Here is a description of a 7-cycle. Let $\lambda \subseteq \Sigma$ be an embedded circle with $\Pi \subseteq \lambda$. This determines a cyclic ordering on $\Pi$ where we index the punctures as $p_1, p_3, p_5, p_7, p_2, p_4, p_6$. Let $l_{13}$ be the segment between $p_1$ and $p_3$ etc. Thus $\lambda = l_{13} \cup l_{35} \cup l_{57} \cup l_{72} \cup l_{24} \cup l_{46} \cup l_{61}$. Let $B_i$ be a regular neighbourhood of $l_{i-2,i} \cup l_{i,i+2}$ with $B_i \cap \Pi = \{p_{i-2}, p_i, p_{i+2}\}$, and let $\gamma_i = \partial B_i$. Thus, $B_i = B(\gamma_i)$. Note that $B_i \cap B_{i+1} = \emptyset$, and so $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$ is a 7-cycle. Any non-adjacent pairs of curves intersect exactly twice. (Figure 1.)

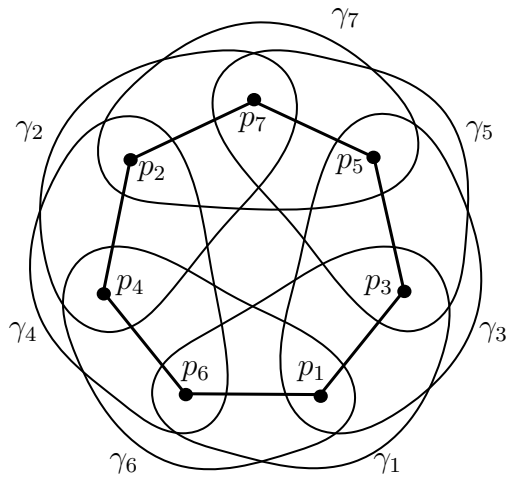


FIGURE 1. A 7-cycle in $\mathcal{G}_{ss}(S_{0,7})$

Now $\mathrm{Map}(S_{0,7})$ acts on $\mathcal{G}_{ss}(S_{0,7})$. The main aim of this section is to show:

**Proposition 3.1.** *There is exactly one 7-cycle in $\mathcal{G}_{ss}(S_{0,7})$ up to the action of $\mathrm{Map}(S_{0,7})$.*

We begin with an analogous statement for 3-sets in a 7-set. Given any set, $\Psi$, let $\Theta = \Theta(\Psi)$ be the graph whose vertex set, $V(\Theta)$, consists of subsets of cardinality 3 in $\Psi$; and whose edge set, $E(\Theta)$, consists of pairs of disjoint such 3-sets.

For the remainder of this section we will assume that $|\Psi| = 7$. In this case it is a connected 4-regular graph on thirty-five vertices. Note also that there is an edge colouring, $\chi : E(\Theta) \longrightarrow \Psi$, given by $\Psi \setminus (P \cup Q) = \{\chi(e)\}$, where $e$ is the edge from $P$ to $Q$.

The following simple observation will be useful:

**Lemma 3.2.** *If $P, Q \in V(\Theta)$ are 2-distant, then $|P \cap Q| = 2$. If $P, Q$ are 3-distant, then $|P \cap Q| = 1$.*

The following is the analogue, in $\Theta$, of the the main result of this section:

**Lemma 3.3.** *If $|\Psi| = 7$, there are no 3-cycles or 5-cycles in $\Theta(\Psi)$. There is exactly one 7-cycle up to the action of* $\mathrm{Sym}(\Psi)$.

*Proof.* The non-existence of 3-cycles is trivial.

Suppose $P_1, P_2, P_3, P_4, P_5$ were a 5-cycle. Then $P_3$ and $P_4$ are both 2-distant from $P_1$, so $|P_1 \cap P_3| = |P_1 \cap P_4| = 2$, and we get the contradiction that $P_3 \cap P_4 \neq \emptyset$.

Writing $\Psi = \{1, 2, 3, 4, 5, 6, 7\}$, there is a 7-cycle in $\Psi$ given by 613–724–135–246–357–461–572. We want to show this is the only one up to the action of $\mathrm{Sym}(\Psi)$.

Suppose, then that $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ is a 7-cycle. Since there is no smaller odd cycle, this must be isometrically embedded in $\Theta$. Suppose that two edges have the same colour, $p$, say. Since the $P_i$ are all distinct, these edges cannot meet or contain adjacent vertices. Thus, up to cyclic reordering, the only possibility for this pair of edges is $P_1, P_2$ and $P_4, P_5$. Now $P_2$ and $P_4$ are 2-distant, and so $|P_2 \cap P_4| = 2$. Now, $P_1$ and $P_5$ are the complements of these sets in $\Psi \setminus \{p\}$, and so we also have $|P_1 \cap P_5| = 2$. But these are 3-distant, so this contradicts Lemma 3.2.

Therefore, each colour occurs exactly once around the cycle. Up to $\mathrm{Sym}(\Psi)$, we can assume they occur in the cyclic order 1234567, starting with the edge $P_1, P_2$. Now consider the sequence $P_1, P_3, P_5, P_7$. We must proceed by replacing 2 by 1, then 4 by 3, then 6 by 5. So we must have started with $P_1$ being 246 (and ended with $P_7$ as 135). But now the whole 7-cycle starting with $P_1$ is completely determined by the colours on the edges. In fact, it must be precisely the cyclic sequence given above. This proves the result. □

We now move on to the proof of Proposition 3.1. We set $\Psi = \Pi$. Recall that we have defined $\pi : C_{ss} \longrightarrow V(\Pi)$ by $\pi(\gamma) = \Pi \cap B(\gamma)$. This extends to a map $\pi : \mathcal{G}_{ss} \longrightarrow \Theta$, sending edges to edges. Composing with $\chi$, we also get a colouring of the edges of $\mathcal{G}_{ss}$, which we also denote by $\chi : E(\mathcal{G}_{ss}) \longrightarrow \Pi$. Note that it is now an immediate consequence that there are no 5-cycles in $\mathcal{G}_{ss}$, as stated earlier.

We need to make a few observations about configurations of curves separated by given distances in $\mathcal{G}_{ss}$.

If $\alpha, \beta \in C_{ss}$ are adjacent, we set $A = A(\alpha, \beta)$ to be (the closure of) $S \setminus (B(\alpha) \cup B(\beta))$. This is an annulus, with $\partial A = \alpha \cup \beta$, and with $A(\alpha, \beta) \cap \Pi = \{p\}$, where $p = \chi(\alpha, \beta)$.

By an *arc* in $A$, we mean an arc $a \subseteq A \setminus \Pi$ with endpoints $\partial a = a \cap \partial A$. We generally regard such an arc as defined up to homotopy in $A \setminus \Pi$, allowing ourselves to slide an endpoint of $a$ in $\partial A$. Up to homotopy, there are exactly three types of arc, depending on whether $a$ meets only $\alpha$, only $\beta$, or both $\alpha$ and $\beta$. We refer to these classes as *$\alpha$-type*, *$\beta$-type* or *crossing* arcs, respectively. Note that an $\alpha$-type arc and a $\beta$-type arc meet (minimally) in exactly two points. (Figure 2.)

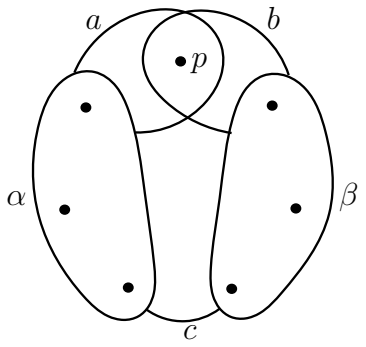


FIGURE 2. $a$: $\alpha$-type arc. $b$: $\beta$-type arc. $c$: crossing arc.

Suppose now that $\beta, \delta \in C_{ss}$ are 2-distant. Then $|\pi(\beta) \cap \pi(\delta)| = 2$, so $\pi(\delta) \setminus \pi(\beta) = \{q\}$ for some $q \in \Pi$. Let $D = D(\delta, \beta)$ be (the closure of) the component of $B(\delta) \setminus B(\beta)$ containing $q$. We claim that $D$ is a bigon:

**Lemma 3.4.** *If $\beta, \delta \in C_{ss}$ are 2-distant, and $D = D(\beta, \delta)$, then $D \cap B(\beta)$ consists of a single arc in $\beta$. (Figure 3.)*

*Proof.* Let $\gamma \in C_{ss}$ be adjacent to both $\beta$ and $\delta$. Note $B(\beta), B(\delta) \subseteq S \setminus B(\gamma)$, and so $D \subseteq A(\beta, \gamma)$ and $D \cap \gamma = \emptyset$. Thus, $\partial D$ can contain

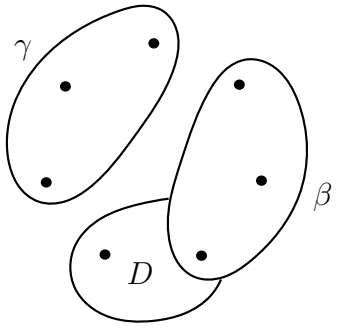


FIGURE 3. The bigon $D$

only $\beta$-type arcs in $A(\beta, \gamma)$. There can only be one of these, so the statement follows easily. □

Suppose now that $\alpha, \beta$ are adjacent and that $\epsilon \in C_{ss}$ is 3-distant from both $\alpha$ and $\beta$. Let $A = A(\alpha, \beta)$ and $p = \chi(\alpha, \beta)$ as before. By Lemma 3.2, $|\pi(\epsilon) \cap \pi(\alpha)| = |\pi(\epsilon) \cap \pi(\beta)| = 1$, and so it follows that $p \in \pi(\epsilon)$. Let $F = F(\alpha, \beta; \epsilon)$ be (the closure of) the component of $A \cap B(\epsilon)$ containing $p$. Note that $F$ must intersect either $\alpha$ or $\beta$, possibly both. (In fact, given that there are only three classes of arc in $A$, one can easily see that $F$ can meet each of $B(\alpha)$ and $B(\beta)$ in at most a single arc, though we won't explicitly need this.)

Now let $\sigma$ be any 7-cycle in $\mathcal{G}_{ss}$.

Suppose that $\alpha, \beta$ is an edge of $\sigma$. Let $\epsilon$ be the vertex of $\sigma$ opposite this edge. Thus, $\epsilon$ is 3-distant from both $\alpha$ and $\beta$, as above. Let $F = F(\alpha, \beta; \epsilon)$, as before.

Suppose that $\beta \cap F \neq \emptyset$. This implies that any $\alpha$-type arc in $A = A(\alpha, \beta)$ must intersect $F$. Let $\gamma$ be the vertex of $\sigma$ adjacent to $\beta$ and distinct from $\alpha$ (so that $\gamma$ and $\epsilon$ are 2-distant). We claim:

**Lemma 3.5.** *Let $\alpha, \beta, \gamma$ be consecutive vertices of $\sigma$. Let $\epsilon$ be the vertex of $\sigma$ opposite the edge $\alpha, \beta$, and let $F = F(\alpha, \beta; \epsilon)$. If $F \cap \beta \neq \emptyset$, then $\iota(\alpha, \gamma) = 2$*

*Proof.* Let $\delta$ be the vertex of $\sigma$ between $\gamma$ and $\epsilon$ (so that $\alpha, \beta, \gamma, \delta, \epsilon$ are consecutive vertices of $\sigma$). Let $D = D(\delta, \beta)$. By Lemma 3.4, this is a bigon; that is, $\partial D = b \cup d$, where $b$ and $d$ are respectively arcs of $\beta$ and $\delta$. Now $F \subseteq B(\epsilon)$, $D \subseteq B(\delta)$ and $B(\epsilon) \cap B(\delta) = \emptyset$, so $F \cap D = \emptyset$. Let $\pi(\delta) \setminus \pi(\beta) = \{q\}$. Note that $q \neq p$, and so $q \in \pi(\alpha) \subseteq B(\alpha)$. Now $d \cap F \subseteq D \cap F = \emptyset$, and so $d \cap A$ contains no $\alpha$-type arcs. Since $d \cap \beta$ are the endpoints of $d$, it cannot contain any $\beta$-type arcs either. Thus,

$d \cap A$ consists only of crossing arcs, of which there must be exactly 2. This means that $|d \cap \alpha| = 2$ (with $d \cap B(\alpha)$ consisting of a single arc). (Figure 4.)

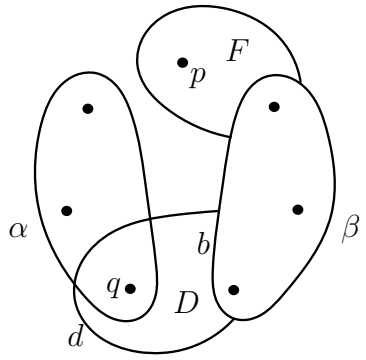


FIGURE 4. $D = D(\delta, \beta)$, $F = F(\alpha, \beta; \epsilon)$

Let $R = B(\beta) \cup D$. Now $R$ is a disc with $R \cap \Pi = \pi(\beta) \cup \{q\} = \pi(\beta) \cup \pi(\alpha)$. Also $R \subseteq B(\beta) \cup B(\delta)$, so $R \cap B(\gamma) = \emptyset$. It follows that $\gamma$ and $\partial R$ are homotopic in $S \setminus \Pi$. In other words, they represent the same element of $C_{ss}$. But now, $\partial R \subseteq \beta \cup d$, so $|\partial R \cap \alpha| = 2$. Thus, $\iota(\alpha, \gamma) \leq 2$, and so, in fact, $\iota(\alpha, \gamma) = 2$ as required. (Figure 5.) □

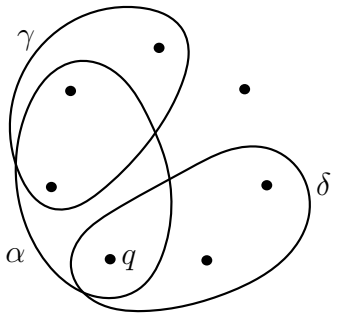


FIGURE 5. The curves $\alpha, \delta, \epsilon$

Now, as already observed, at most one of $F \cap \alpha$ or $F \cap \beta$ can be empty. If $F \cap \beta = \emptyset$, we refer to $\beta$ as a *bad* endpoint of the edge $\alpha, \beta$. We say that a vertex of $\sigma$ is *bad* if it a bad endpoint of both incident edges of $\sigma$. Thus, Lemma 3.5 tells us that if $\beta$ is not bad, then the two vertices

adjacent to $\beta$ in $\sigma$ correspond to curves which intersect twice. (In fact, Proposition 3.1 will retrospectively rule out bad vertices altogether.) In Figure 4 above, $\beta$ would be a bad vertex of the edge $\alpha, \beta$.

*Proof of Proposition 3.1.* Let $\Sigma$ be a 7-cycle.

Now no two bad vertices of $\sigma$ can be adjacent. It follows that there can be at most three bad vertices in total. In fact, we can index the vertices of $\sigma$ consecutively (mod 7) as $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$, so that none of $\gamma_1$, $\gamma_3$, $\gamma_4$ or $\gamma_6$ are bad. It then follows that $\iota(\gamma_7, \gamma_2) = \iota(\gamma_2, \gamma_4) = \iota(\gamma_3, \gamma_5) = \iota(\gamma_5, \gamma_7) = 2$.

We write $B_i = B(\gamma_i)$, and label the points of $\Pi$ as $p_i$, so that $\pi(\gamma_i) = B_i \cap \Pi = \{p_{i-2}, p_i, p_{i+2}\}$.

Consider first the discs $B_2, B_3, B_4, B_5$. We have $B_2 \cap B_3 = B_3 \cap B_4 = B_4 \cap B_5 = \emptyset$. Let $A = A(\gamma_3, \gamma_4)$. Since $\iota(\gamma_3, \gamma_5) = 2$, we see that $\gamma_5 \cap A$ consists of a single $\gamma_3$-type arc. Similarly, $\gamma_2 \cap A$ consists of a single $\gamma_4$-type arc. It follows that $\iota(\gamma_2, \gamma_5) = 2$. Thus, $B_2 \cap B_5$ is a disc with $B_4 \cap B_5 \cap \Pi = \{p_7\}$. Now $B_3 \cap B_5$ and $B_2 \cap B_4$ are also discs, with $B_3 \cap B_5 \cap \Pi = \{p_3, p_5\}$ and $B_2 \cap B_4 \cap \Pi = \{p_2, p_4\}$. We can therefore find an arc $l$ with $\Pi \subseteq l$, with endpoints $p_1$ and $p_6$ and with the points $p_1, p_3, p_5, p_7, p_2, p_4, p_6$ occurring in this order along $l$, and so that $B_3$, $B_5$, $B_2$, $B_4$ are respectively regular neighbourhoods of $l_{13} \cup l_{35}$, $l_{35} \cup l_{57}$, $l_{72} \cup l_{24}$ and $l_{24} \cup l_{46}$, where we have cut $l$ into six arcs, $l = l_{13} \cup l_{35} \cup l_{57} \cup l_{72} \cup l_{24} \cup l_{46}$ connecting the points of $\Pi$. Note in particular, that $R = B_2 \cup B_5$ is a disc with $R \cap \Pi = \{p_3, p_5, p_7, p_2, p_4\}$. Also $B_2 \cap B_5$ is a disc with $B_2 \cap B_5 \cap \Pi = \{p_7\}$. (Figure 6.)

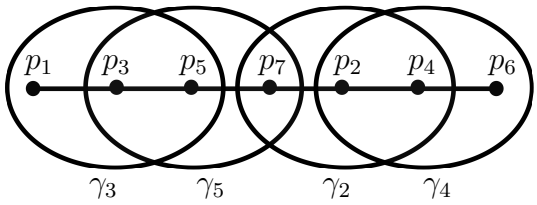


FIGURE 6. The curves $\gamma_2, \gamma_3, \gamma_4, \gamma_5$

Now consider $B_7$. We have $B_7 \cap \Pi = \{p_5, p_7, p_2\}$. Now $\iota(\gamma_5, \gamma_7) = 2$, so $\gamma_7 \cap B_5$ consists of a single arc separating $p_3$ from $p_5$ and $p_7$ in $B_5$. We can therefore realise it so that it is disjoint from $B_2 \cap B_5$. In fact, we can take this arc to meet $l$ just once, in a point of the segment $l_{35}$. Similarly, we can realise $\gamma_7 \cap B_2$ as a single arc, also disjoint from $B_2 \cap B_5$, and meeting $l$ in a single point of $l_{24}$. In this way, $\gamma_7 \setminus R$ consists of exactly two arcs. Since $(B_7 \setminus R) \cap \Pi = \emptyset$, each of these arcs

can be homotoped into $\partial R$ in $S \setminus \Pi$, fixing their endpoints. We can therefore realise $B_7$ as a regular neighbourhood of $l_{57} \cup l_{72}$.

Now let $l_{61}$ be any arc in $S$ meeting $l$ exactly at their common endpoints. Thus $\lambda = l \cup l_{61}$ is a circle containing $\Pi$. Now the homotopy classes of $\gamma_1$ and $\gamma_6$ are determined as $\partial(B_7 \cup B_2)$ and $\partial(B_7 \cup B_4)$. We can therefore realise $B_1$ and $B_6$ respectively as regular neighbourhoods of $l_{46} \cup l_{61}$ and of $l_{61} \cup l_{13}$.

We are therefore exactly in the situation of the example of a 7-cycle described earlier (as in Figure 1).

This proves Proposition 3.1. □

We note the following immediate consequence.

**Lemma 3.6.** *Any automorphism of $\mathcal{G}_{ss}(S_{0,7})$ preserves the set of surrounding pairs.*

*Proof.* In view of Proposition 3.1, we see that $\alpha, \beta$ form a surrounding pair if and only if they are 2-distant vertices in some 7-cycle in $\mathcal{G}_{ss}$. Note that this is determined just by the structure of $\mathcal{G}_{ss}$. □

## 4. Rigidity for the 7-holed sphere

Recall that $C_0 = C_0(S_{0,7})$ is the set of curves in $S \setminus \Pi$ which bound a disc containing exactly two points of $\Pi$. If $\omega \in C_0$, this disc is $B(\omega)$ and $\pi(\omega) = B(\omega) \cap \Pi$. If $\omega \in C_0$ and $\alpha \in C_{ss}$, then $\omega < \alpha$ means that $B(\omega) \subseteq B(\alpha)$.

By an *$\omega$-arc* we mean an arc, $a$, in $S$ meeting $B(\omega)$ at one endpoint (the *initial* endpoint) and $\Pi$ at the other (*terminal* endpoint). We regard $a$ as being defined up to homotopy relative to $\Pi$, fixing the terminal endpoint, and allowing the initial endpoint to slide along $\omega$. Note that $a$ determines an element $\alpha \in C_{ss}$, with $\omega < \alpha$, so that $B(\alpha)$ is a regular neighbourhood of $B(\omega) \cup a$. In fact, every $\alpha \in C_{ss}$ with $\omega < \alpha$ arises in this way.

If $\alpha, \beta \in C_{ss}$ is a surrounding pair, then there is a unique $\omega \in C_0$ with $\omega < \alpha$ and $\omega < \beta$. In fact, $\omega = \partial(B(\alpha) \cap B(\beta))$. Note that $\alpha, \beta$ correspond to disjoint $\omega$-arcs, $a, b$ (i.e. we can realise $a, b$ to be disjoint). We say that $\alpha, \beta$ *surround* $\omega$. (Figure 7.)

Recall that a *surrounding triple* consists of three curves, $\alpha, \beta, \gamma \in C_{ss}$, such that $\{\alpha, \beta\}$, $\{\beta, \gamma\}$ and $\{\gamma, \alpha\}$ are all surrounding pairs, and such that there is some $\omega \in C_0$ with $\omega < \alpha, \beta, \gamma$. In this case, $\alpha, \beta, \gamma$ correspond to three pairwise disjoint $\omega$-arcs, $a, b, c$.

Suppose we just know that $\alpha, \beta, \gamma \in C_{ss}$ are such that any pair form a surrounding pair. Then $\pi(\alpha), \pi(\beta), \pi(\gamma)$ pairwise intersect in sets of two elements. It follows easily that $\pi(\alpha) \cup \pi(\beta) \cup \pi(\gamma)$ has either 4 or

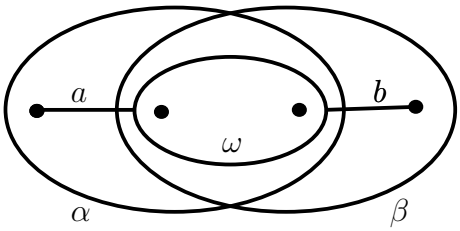


FIGURE 7. A surrounding pair, $\alpha, \beta$

5 elements. In the former case, $|\pi(\alpha) \cap \pi(\beta) \cap \pi(\gamma)| = 1$, and one sees easily that there is a curve in $C_{ss}$ (namely $\partial(B(\alpha) \cup B(\beta) \cup B(\gamma))$), disjoint from each of $\alpha, \beta, \gamma$. In the latter case, $|\pi(\alpha) \cap \pi(\beta) \cap \pi(\gamma)| = 2$, and one sees easily that $\alpha, \beta, \gamma$ form a surrounding triple. We deduce:

**Lemma 4.1.** *Suppose $\alpha, \beta, \gamma \in C_{ss}$, then $\alpha, \beta, \gamma$ form a surrounding triple if and only if any pair of them form a surrounding pair and there is no curve in $C_{ss}$ which is disjoint from each of $\alpha, \beta, \gamma$.*

In view of Lemma 3.6, we see that we can recognise when three elements of $C_{ss}$ form a surrounding triple.

Let $\mathcal{H}$ be the graph whose vertex set is the set of all surrounding pairs in $C_{ss}$, and where two vertices are deemed adjacent if the union of the two pairs is a surrounding triple. Given any $\omega \in C_0$, let $\mathcal{H}(\omega)$ be the full subgraph whose vertex set consists of those surrounding pairs which surround $\omega$. Now adjacent vertices of $\mathcal{H}$ determine the same element of $C_0$, and so $\mathcal{H}(\omega)$ is a union of components of $\mathcal{H}$. In fact:

**Lemma 4.2.** *If $\omega \in C_0$, then the graph $\mathcal{H}(\omega)$ is connected.*

*Proof.* This is best seen in terms of $\omega$-arcs. Recall that a vertex of $\mathcal{H}(\omega)$ corresponds to a pair of disjoint $\omega$-arcs, and an edge of $\mathcal{H}(\omega)$ corresponds to a triple of pairwise disjoint $\omega$-arcs.

Suppose that $a, b, c, d$ are $\omega$-arcs with $a \cap b = \emptyset$ and $c \cap d = \emptyset$. We realise them in general position in $S$. (They do not need to have minimal intersection in their homotopy classes.) We aim to connect the vertices $a, b$ and $c, d$ by a path in $\mathcal{H}(\omega)$. Write $I(a, b; c, d) = (a \cup b) \cap (c \cup d) \setminus \Pi$ for the set of interior intersection points. We proceed by induction on $|I(a, b; c, d)|$.

The case where that $I(a, b; c, d) = \emptyset$ is elementary, so we assume there is some $x \in I(a, b; c, d)$. After permuting $a, b$ and $c, d$, we can assume that $x \in a \cup c$, and that $x$ is the first intersection point along $c$; that is, the initial segment $e$, of $c$ ending at $x$, meets $a \cup b$ only

at $x$. Let $f$ be the initial segment of $a$ ending at $x$, and set $a' = (a \setminus f) \cup e$. We can move $a'$ slightly so that $a \cap a'$ meet precisely in their terminal points, while retaining disjointness from $b$. Now, we can easily find an $\omega$-arc, $e$, disjoint from each of $b, a, a'$, so $a, b, e$ and $a', b, e$ correspond to surrounding triples. It follows that $a, b$ and $a', b$ correspond to surrounding pairs in the same component of $\mathcal{H}(\omega)$. Note that $|I(a', b; c, d)| < |I(a, b; c, d)|$. We therefore replace $a, b$ by $a', b$ and proceed inductively. □

We deduce:

**Lemma 4.3.** *There is a natural bijective correspondence between the elements of $C_0$ and the connected components of $\mathcal{H}$, such that if $\omega \in C_0$ and $\alpha \in C_{ss}$, then $\omega < \alpha$ if and only if $\alpha$ occurs as a curve in a surrounding pair of some vertex of the corresponding component of $\mathcal{H}$.*

Since $\mathcal{H}$ can be constructed out of $\mathcal{G}_{ss}$, we see that we can also reconstruct $C_0$ and the relation $<$ between $C_0$ and $C_{ss}$ out of $\mathcal{G}_{ss}$.

We can also recognise disjointness. If $\omega \in C_0$ and $\alpha \in C_{ss}$, then $\omega$ and $\alpha$ are disjoint if and only if there is some $\beta \in C_{ss}$ such that $\omega < \beta$ and $\alpha, \beta$ are either equal or disjoint. Similarly, if $\omega, \omega' \in C_0$, then $\omega$ and $\omega'$ are disjoint if and only if there are disjoint curves $\alpha, \alpha' \in C_{ss}$ with $\omega < \alpha$ and $\omega' < \alpha'$.

We can therefore reconstruct the graph $\mathcal{G}(S_{0,7}) = \mathcal{G}_s(S_{0,7})$ from the graph $\mathcal{G}_{ss}(S_{0,7})$. It follows that any automorphism of the former extends to an automorphism of the latter. But we know that $\mathcal{G}(S_{0,7})$ is rigid by Theorem 1 of [Ko], so we have shown:

**Proposition 4.4.** *The graph $\mathcal{G}_{ss}(S_{0,7})$ is rigid.*

## 5. Multicurves

In this section, we will explain how to identify classes of multicurves from the structure of the strongly separating curve graph. We assume that $\Sigma$ is a compact orientable surface, with boundary $\partial\Sigma$, which we view as a (possibly empty) set of curves. (Thus, $S_{0,7}$ reverts to being a bona fide 7-holed sphere.) We will assume that $\xi(\Sigma) \geq 4$.

As before, we say that we can *recognise* a property of a collection of curves in $C_{ss} = C_{ss}(\Sigma)$ if this property preserved under any automorphism of $\mathcal{G}_{ss}(\Sigma)$. In other words, it can be seen just in terms of the graph structure. We similarly say that we can "tell" if a given property holds. We also say that another graph can be "constructed" (from $\mathcal{G}_{ss}$) etc.

Recall that $C_1 \subseteq C_{ss}$ is the set of $\alpha \in C_{ss}$ for which $B(\alpha)$ has complexity 1, that is either an $S_{0,4}$ or an $S_{1,1}$.

Suppose that $\alpha, \beta, \gamma \in C_{ss}$. We can tell if $\alpha$ separates $\beta$ from $\gamma$. (It is equivalent to saying that any curve in $C_{ss}$ is disjoint from $\alpha$ must also be disjoint from either $\beta$ or $\gamma$.) Therefore, we can recognise elements of $C_1$: a curve in $C_{ss}$ lies in $C_1$ if and only if it does not separate any two other elements of $C_{ss}$.

By a *multicurve*, $\tau$, in $C_{ss}$, we mean a non-empty set of pairwise disjoint curves in $C_{ss}$. We will sometimes abuse notation by regarding $\tau$ as a subset of $\Sigma$. We claim that we can identify $\tau$ up to the action of $\mathrm{Map}(\Sigma)$. This is equivalent to saying that we can recognise the topological types of each of the components of $\Sigma \setminus \tau$ together with the elements of $\tau$ which a bound a given component.

Let $X$ be a component of $\Sigma \setminus \tau$. Write $\partial X \subseteq \tau \cup \partial\Sigma$ for its intrinsic boundary, and $\partial_\Sigma X = \tau \cap \partial X$ for the relative boundary. We write $p(X) = |\partial X|$, $q(X) = |\partial_\Sigma X|$ and $g(X)$ for the genus of $X$. Write $C_{ss}(\Sigma, X)$ for the set of elements of $C_{ss}(\Sigma) \setminus \tau$ contained in $X$. Note that $C_{ss}(\Sigma, X)$ is either empty or infinite.

**Definition.** We say that a complementary component, $X$, is *large* if $C_{ss}(\Sigma, X)$ is infinite.

Thus, if $X$ is not large if and only if is either an $S_{0,3}$ with $q(X) \geq 2$, or else has the form $B(\alpha)$ for some $\alpha \in C_1(\Sigma)$.

Note that if $\alpha \in C_{ss}$, then $\alpha \in C_{ss}(\Sigma, X)$ for some large component, $X$, if and only if it does not lie in $\tau$ and is disjoint from $\tau$ (i.e. is disjoint from each element of $\tau$). Given two such curves at $\alpha, \beta \in C_{ss}$ then $\alpha, \beta$ lie in the same set $C_{ss}(\Sigma, X)$ if and only if they are not separated by any element of $\tau$. Thus, from $\tau$ we can identify the collection of sets $C_{ss}(\Sigma, X)$ which arise from the large components, $X$, of $\Sigma \setminus \tau$. We next want to recognise the topological type of such $X$.

To this end, we define a *chain* in $C_{ss}$ to be a sequence, $\gamma_0, \gamma_1, \ldots, \gamma_n$, of disjoint curves such that $\gamma_j$ separates $\gamma_i$ from $\gamma_k$ whenever $i < j < k$.

Suppose that $X$ is a component of $\Sigma \setminus \tau$ with $q(X) \geq 2$. Choose any distinct $\alpha, \beta \in \partial_\Sigma X$, and let $n$ be maximal so that there is a chain $\alpha = \gamma_0, \ldots, \gamma_n = \beta$ in $C_{ss}$ (so that $\gamma_i \in C_{ss}(\Sigma, X)$ for all $i \neq 0, n$). Each component of $X \setminus \bigcup_i \gamma_i$ is either an $S_{0,3}$ or an $S_{1,2}$. (Figure 8.) Moreover, we can tell if the component between $\gamma_i$ and $\gamma_{i+1}$ is an $S_{1,2}$, since in that case, there will be some $\delta \in C_{ss}(\Sigma, X)$ such that $\gamma_i$ separates $\delta$ from $\alpha$, and $\gamma_{i+1}$ separates $\delta$ from $\beta$. Thus, we know the number, $m$, of such $S_{1,2}$ components. We see that $g(X) = m$ and $p(X) = n - m + 2$, so we have determined the type of $X$ in this case.

Now suppose that $q(X) = 1$, and that $X$ is large (equivalently, not an $S_{0,4}$ nor an $S_{1,1}$). Write $\partial_\Sigma X = \{\alpha\}$. Suppose that $\beta \in C_{ss}(\Sigma, X) \cap C_1(\Sigma)$ (which we can recognise). Let $Y = X \setminus B(\beta)$, so that $Y$ is a

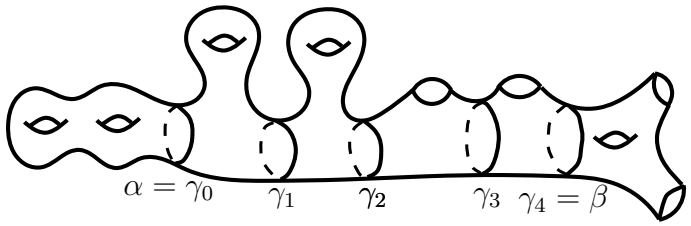


FIGURE 8. Here, $X$ is the subsurface bounded by $\alpha \cup \beta$

component of $\Sigma \backslash (\tau \cup \beta)$, and $\partial_\Sigma Y = \{\alpha, \beta\}$. By the previous paragraph, we know the type of $Y$ (given $\beta$). Therefore we know the collection of types of all such $Y$ which can arise in this way. Given that $B(\beta)$ is either an $S_{0,4}$ or an $S_{1,1}$, there are at most two such types. If $\xi(X) \geq 4$, we now see easily that this data determines the topological type of $X$. (Note that if there is only one type for $Y$, then either $g(X) = 0$ or $p(X) \leq 3$. If there are two types, then $p(X) \geq 4$.) However, if $\xi(X) \leq 3$, then the data does not allow us to distinguish the pairs $\{S_{0,5}, S_{1,2}\}$ or $\{S_{0,6}, S_{1,3}\}$. (For if $X$ is an $S_{0,5}$ or $S_{1,2}$, then $Y$ must be an $S_{0,3}$; and if $X$ is an $S_{0,6}$ or $S_{1,3}$, then $Y$ must be an $S_{0,4}$.)

Now suppose that $\delta \in C_1(\Sigma)$. Let $Z = \Sigma \setminus B(\delta)$. Suppose, that $\xi(\Sigma) \geq 6$, and that $\Sigma \neq S_{0,7}$. We see that $Z \neq S_{0,5}, S_{0,6}, S_{1,2}, S_{1,3}$. Thus, by the previous paragraph, we can determine the type of $Z$. Since this holds for all elements of $C_1(\Sigma)$, we can now easily determine the topological type of $\Sigma$, and also tell whether an element of $C_1(\Sigma)$ bounds an $S_{0,4}$ or an $S_{1,1}$.

Retrospectively, we can now go back to the earlier set-up and distinguish an $S_{0,5}$ from an $S_{1,2}$, or an $S_{0,6}$ from an $S_{1,3}$ in the complement of $\tau$. We therefore now know the types of all large components. From this one can easily determine $\tau$ up to the action of $\mathrm{Map}(\Sigma)$.

In summary, we have shown:

**Lemma 5.1.** *Suppose $\xi(\Sigma) \geq 6$. Suppose that $\tau, \tau' \subseteq C_{ss}(\Sigma)$ are two multicurves, and that there is an automorphism of $\mathcal{G}_{ss}(\Sigma)$ taking $\tau$ to $\tau'$. Then there is an element of* $\mathrm{Map}(\Sigma)$ *taking $\tau$ to $\tau'$.*

Given a component, $X$, of $\Sigma \setminus \tau$, set $C_{ss}(X)$ and $\mathcal{G}_{ss}(X)$ as defined intrinsically to $X$. Thus, $\mathcal{G}_{ss}(X)$ is the full subgraph of $\mathcal{G}_{ss}(\Sigma, X)$ with vertex set $C_{ss}(X)$. Note that we can tell whether a curve $\gamma \in C_{ss}(\Sigma, X)$ lies in $C_{ss}(X)$ (since it does not bound an $S_{0,3}$ component of $\Sigma \backslash (\tau \cup \gamma)$). Thus, we can construct $\mathcal{G}_{ss}(X)$ out of $\mathcal{G}_{ss}(\Sigma)$, given $\tau$.

Note that the above encompasses all cases where $g(\Sigma) + p(\Sigma) \geq 7$.

We remark that we have also proven the following:

**Proposition 5.2.** *Suppose that $\Sigma, \Sigma'$ are compact surfaces with $\mathcal{G}_{ss}(\Sigma)$ isomorphic to $\mathcal{G}_{ss}(\Sigma')$. If $\xi(\Sigma) \geq 6$, then $\Sigma' = \Sigma$.*

Of course, this leaves open a number of cases, which we will not address here.

## 6. Rigidity of other surfaces

In this section we will prove Theorem 1.1, except in the case where $\Sigma = S_{0,8}$. We assume that $g(\Sigma) + p(\Sigma) \geq 7$. We will split into three cases: First, $p(\Sigma) \geq 5$, $\Sigma \neq S_{0,8}$; second, $p(\Sigma) \leq 4$, $g(\Sigma) \geq 4$; and finally $\Sigma \in \{S_{3,4}, S_{4,3}, S_{4,4}\}$.

Recall that for $\alpha, \beta \in C_{ss}(\Sigma)$, $\alpha < \beta$ means that $\alpha \neq \beta$ and $B(\alpha) \subseteq B(\beta)$.

First, consider the case where $p(\Sigma) \geq 5$. In this case, we will also assume that $\Sigma \neq S_{0,8}$.

We begin by giving a criterion for recognising such pairs (as defined in Section 2).

**Lemma 6.1.** *Suppose that $\alpha, \beta \in C_{ss}(\Sigma)$. Then $\alpha, \beta$ is a surrounding pair if and only each of the following three conditions holds:*

*(S1): $B(\alpha)$ and $B(\beta)$ are both $S_{0,4}$'s,*
*(S2): There is some multicurve, $\tau$, in $\Sigma$ such that $\alpha, \beta$ both lie in some component, $X$, of $\Sigma \setminus \tau$, of type $S_{0,7}$, and*
*(S3): $\alpha, \beta$ form a surrounding pair intrinsically in $X$.*

(To make sense of (S3), note that necessarily $\alpha, \beta \in C_{ss}(X)$.)

*Proof.* First, suppose that $\alpha, \beta$ is a surrounding pair in $\Sigma$. Note that $Y = \Sigma \setminus (B(\alpha) \cup B(\beta))$ satisfies $g(Y) = g(\Sigma)$ and $p(Y) = p(\Sigma) - 3$. In particular, $g(Y) + p(Y) \geq 4$ and $Y$ is not an $S_{0,5}$ (since $\Sigma \neq S_{0,8}$). Thus, we can find a multicurve, $\tau \subseteq C_{ss}(Y)$ so that $X = Y \setminus \bigcup_{\gamma \in \tau} B'(\gamma)$ is an $S_{0,7}$. Here, $B'(\gamma)$ is the component of $\Sigma \setminus \gamma$ not containing $\alpha, \beta$. (It might not be the lower-complexity component.) We can, of course, assume the $B'(\gamma)$ to be disjoint. Now $B(\alpha), B(\beta) \subseteq X$, and we see that $\alpha, \beta$ is a surrounding pair in $X$. Thus, $\alpha, \beta$ satisfies (S1)–(S3).

Conversely, if $\alpha, \beta$ satisfies (S1)–(S3), then again we must have $B(\alpha), B(\beta) \subseteq X$ and so $B(\alpha) \cap B(\beta)$ is an $S_{0,3}$, so $\alpha, \beta$ is a surrounding pair in $\Sigma$. $\square$

**Lemma 6.2.** *If $g(\Sigma) + p(\Sigma) \geq 7$, $p(\Sigma) \geq 5$, and $\Sigma \neq S_{0,8}$, then the collection of surrounding pairs is invariant under any automorphism of $\mathcal{G}_{ss}(\Sigma)$.*

*Proof.* By Lemma 3.1, it's enough to show that properties (S1)–(S3) are all recognisable in terms of $\mathcal{G}_{ss}(\Sigma)$. We can certainly recognise a multicurve, $\tau$, in $C_{ss}(\Sigma)$, and by Lemma 5.1, we can tell if $\alpha, \beta$ lie in a component $X$ of $\Sigma \setminus \tau$ of type $S_{0,7}$. As explained at the end of Section 5, we can also construct $\mathcal{G}_{ss}(X)$. By Lemma 3.6, we can tell if $\alpha, \beta$ form a surrounding pair intrinsically to $X$. □

The class of surrounding triples (as defined in Section 2) is also recognisable in $\mathcal{G}_{ss}(\Sigma)$:

**Lemma 6.3.** *If $g(\Sigma) + p(\Sigma) \geq 7$, $p(\Sigma) \geq 5$ and $\Sigma \neq S_{0,8}$, then the collection of surrounding triples is invariant under any automorphism of $\mathcal{G}_{ss}(\Sigma)$.*

*Proof.* We first make the we make the following general observation, (which holds for any surface $\Sigma$). Suppose that $B_1, B_2, B_3 \subseteq \Sigma$ are connected subsurfaces (in general position) such that the $\partial B_i$ are all connected, and such that $|\partial B_i \cap \partial B_j| = 2$ whenever $i \neq j$. If $B_1 \cap B_2 \cap B_3$ and $\Sigma \setminus (B_1 \cup B_2 \cup B_3)$ are both nonempty, then they are also both connected with connected boundary. This is a simple exercise on noting that a regular neighbourhood of $\partial B_1 \cup \partial B_2 \cup \partial B_3$ is an $S_{0,8}$.

Given this, we can now recognise a surrounding triple, as a triple $\alpha, \beta, \gamma$, with each pair forming a surrounding pair, and such that there is a fourth curve $\delta$ disjoint from $\alpha, \beta, \gamma$ and such that $\alpha, \beta, \gamma$ all lie in an $S_{0,6}$ component of $\Sigma \setminus \delta$. Note that this implies that $B(\alpha) \cap B(\beta) \cap B(\gamma)$ must be non-empty (it must contain a boundary component of $\Sigma$). It now follows easily from the previous paragraph that $B(\alpha) \cap B(\beta) \cap B(\gamma)$ is in fact an $S_{0,3}$, and so we can set $\omega$ to be its relative boundary in $\Sigma$. □

*Proof of Theorem 1.1 when $p(\Sigma) \geq 5$.* The remainder of the proof now follows exactly as for $S_{0,7}$. We define the graphs $\mathcal{H}$ and $\mathcal{H}(\omega)$ in the same way. It is sufficient to show that $\mathcal{H}(\omega)$ is connected. The argument follows exactly as with that of Lemma 4.2. We can define an *$\omega$-arc* to be an arc connecting $B(\omega)$ to $\partial\Sigma \setminus B(\omega)$. Taking a regular neighbourhood of $B(\omega) \cup a \cup \epsilon$, where $\epsilon$ is the boundary component, we get an $S_{0,4}$, namely $B(\alpha)$, where $\alpha = \partial_\Sigma B(\alpha) \in C_{ss}$, so that $\omega < \alpha$. A surrounding pair corresponds to a disjoint pair of $\omega$-arcs, terminating in distinct boundary components. Similarly, a surrounding triple corresponds to a disjoint triple of $\omega$-arcs to distinct boundary components. The we can now copy the argument of Lemma 4.2 (after collapsing each component of $\partial\Sigma \setminus B(\omega)$ to a point). This allows us to reconstruct $\mathcal{G}_s(\Sigma)$, and so, using [Ko, BrM, Ki], we see that $\mathcal{G}_{ss}(\Sigma)$ is rigid in these cases. □

We now move on to the cases where $p(\Sigma) \leq 4$. We can assume that $p(\Sigma) \geq 2$ (otherwise $\mathcal{G}_{ss}(\Sigma) = \mathcal{G}_s(\Sigma)$, and we are covered by [Ko, BrM, Ki]). Note that, $g(\Sigma) \geq 3$ in these cases.

We will use the following construction. Let $\mathcal{T}(\Sigma)$ be the graph whose vertex set consists of those $\alpha \in C_1(\Sigma)$ for which $B(\alpha)$ is an $S_{1,1}$, and where $\alpha, \beta$ are deemed adjacent if $B(\alpha) \cap B(\beta) = \emptyset$. (This is a full subcomplex of $\mathcal{G}_{ss}(\Sigma)$.) We note:

**Lemma 6.4.** *If $g(\Sigma) \geq 3$, then $\mathcal{T}(\Sigma)$ is connected.*

*Proof.* Suppose $\alpha, \beta$ are vertices of $\mathcal{T}(\Sigma)$. Since the separating curve graph of a closed surface of genus at least 3 is connected, we can connect $\alpha, \beta$ by a vertex path $\alpha = \gamma_0, \gamma_1, \ldots, \gamma_n = \beta$ in $\mathcal{G}_s(\Sigma)$, so that no complementary component of any $\gamma_i$ is planar. (To see this, just cap off the boundary components of $\Sigma$ by discs.) Taking $n$ to be minimal, we see that $\gamma_{i-1}$ must intersect $\gamma_{i+1}$ for all $i \neq 0, n$. Let $\delta_i$ be a vertex of $\mathcal{T}(\Sigma)$ contained in the component of $\Sigma \setminus \gamma_i$ not containing $\gamma_{i-1}, \gamma_{i+1}$. We see that $\alpha, \delta_1, \ldots, \delta_{n-1}, \beta$ is a path in $\mathcal{T}(\Sigma)$ connecting $\alpha$ to $\beta$. □

Let $\hat{\mathcal{T}}(\Sigma)$ be the flag simplicial complex with 1-skeleton $\mathcal{T}(\Sigma)$, so that every complete subgraph of $\mathcal{T}(\Sigma)$ is contained in a simplex of $\hat{\mathcal{T}}(\Sigma)$. Given $n \geq 1$, let $\mathcal{S}_n(\Sigma)$ be the graph whose vertex set consists of $n$-simplices of $\hat{\mathcal{T}}(\Sigma)$, and where two such simplices are deemed adjacent if they have a common $(n-1)$-face.

**Lemma 6.5.** *If $g(\Sigma) \geq n + 2$, then $\mathcal{S}_n(\Sigma)$ is connected.*

*Proof.* If $0 \leq m \leq n - 2$, then the link of any $m$-simplex in $\hat{\mathcal{T}}(\Sigma)$ is isomorphic to $\mathcal{T}(\Sigma')$, where $\Sigma'$ is obtained by removing $m + 1$ disjoint copies of $S_{1,1}$ from $\Sigma$. Thus $g(\Sigma') = g(\Sigma) - m - 1 \geq 3$, and so this is connected by Lemma 6.4. Since $\hat{\mathcal{T}}(\Sigma)$ is itself connected, the statement now follows easily. □

We now consider the case where $g(\Sigma) \geq 5$. (This will cover all cases with $p(\Sigma) \leq 4$, except $S_{3,4}$, $S_{4,3}$ and $S_{4,4}$, which we discuss later.) For this, we need to modify the definition of a surrounding pair:

**Definition.** (When $p(\Sigma) \leq 4$ and $g(\Sigma) \geq 5$.) A *surrounding pair* is a pair of curves, $\alpha, \beta \in C_{ss}(\Sigma)$, such that $B(\alpha), B(\beta)$ are both $S_{1,3}$'s and such that $B(\alpha) \cap B(\beta)$ is an $S_{0,3}$.

This implies that $|\partial B(\alpha) \cap \partial B(\beta)| = 2$. Again, there is a unique $\omega \in C_0(\Sigma)$ with $\omega < \alpha, \beta$, namely $\omega = \partial(B(\alpha) \cap B(\beta))$. This property is also recognisable:

**Lemma 6.6.** *If $p(\Sigma) \leq 4$ and $g(\Sigma) \geq 5$, then the collection of surrounding pairs is invariant under any automorphism of $\mathcal{G}_{ss}(\Sigma)$.*

*Proof.* The argument follows exactly as with Lemma 4.2. The criterion of Lemma 6.1 still holds, except that $B(\alpha), B(\beta)$ are now both $S_{1,3}$'s instead of $S_{0,4}$'s. Also, in verifying (S1)–(S3), the multicurve $\tau$ will include curves in $B(\alpha)$ and $B(\beta)$ which bound $S_{1,1}$'s. □

We also define a *surrounding triple* the same way as before (modulo the definition of a surrounding pair). Given Lemma 6.6, we see easily that $\alpha, \beta, \gamma$ form a surrounding triple if and only if they pairwise form surrounding pairs, and there is some $\delta \in C_{ss}(\Sigma)$ such that $\alpha, \beta, \gamma$ all lie in an $S_{3,3}$ component of $\Sigma \setminus \delta$.

*Proof of Theorem 1.1, when $p(\Sigma) \leq 4$ and $g(\Sigma) \geq 5$.* We now proceed, as usual, to define the graphs $\mathcal{H}$ and $\mathcal{H}(\omega)$ for $\omega \in C_0$. We claim that $\mathcal{H}(\omega)$ is connected. This is a bit more involved in this case.

Suppose that $\alpha \in C_{ss}(\Sigma)$ with $B(\alpha)$ an $S_{1,3}$ and with $B(\omega) \subseteq B(\alpha)$. Let $\epsilon \in C_{ss}(\Sigma)$ be any curve with $B(\epsilon)$ an $S_{1,1}$ with $B(\epsilon) \subseteq B(\alpha) \setminus B(\omega)$. Thus, $B(\alpha) \setminus (B(\omega) \cup B(\epsilon))$ is an $S_{0,3}$, so there is (up to homotopy) a unique arc, $a$, in $B(\alpha)$ from $B(\omega)$ to $B(\epsilon)$, meeting $B(\omega)$ and $B(\epsilon)$ precisely at its endpoints. Note that $B(\alpha)$ is a regular neighbourhood of $B(\omega) \cup a \cup B(\epsilon)$. Conversely, given any $\epsilon \in C_{ss}(\Sigma)$ with $B(\epsilon)$ an $S_{1,1}$ disjoint from $B(\omega)$, we can obtain such an $\alpha$ as the boundary of a regular neighbourhood of $B(\omega) \cup a \cup B(\epsilon)$. (Of course, such a representation of $\alpha$ is not unique, but that will not matter.)

Now, a vertex of $\mathcal{H}(\omega)$ arises from a pair of disjoint such curves, $\epsilon, \eta$, and disjoint arcs, $a, b$, connecting $B(\omega)$ respectively to $B(\epsilon)$ and $B(\eta)$, in $\Sigma \setminus (B(\omega) \cup B(\epsilon) \cup B(\eta))$. Similarly, a surrounding triple arises from three disjoint such curves, $\epsilon, \eta, \zeta$, and three disjoint arcs, $a, b, c$ in the complement of $B(\omega) \cup B(\epsilon) \cup B(\eta) \cup B(\zeta)$.

Suppose we fix $\epsilon, \eta, \zeta$, and let $\mathcal{H}(\omega; \epsilon, \eta, \zeta)$ be the full subgraph of $\mathcal{H}(\omega)$, where all the vertices arise (as above) from some pair of curves in $\{\epsilon, \eta, \zeta\}$. Now $\mathcal{H}(\omega; \epsilon, \eta, \zeta)$ is connected. This can be seen by the same argument as in the previous case (applied to the surface $\Sigma \setminus (B(\epsilon) \cup B(\eta) \cup B(\zeta))$, where the notion of "surrounding pair" reverts to the previous case, as defined in Section 2).

Now if $\epsilon, \eta, \zeta, \theta$ are all disjoint such curves, we can connect $B(\omega)$ to each of $B(\epsilon), B(\eta), B(\zeta), B(\theta)$ by disjoint arcs in the complement of $B(\omega) \cup B(\epsilon) \cup B(\eta) \cup B(\zeta) \cup B(\theta)$. It then follows that $\mathcal{H}(\omega; \epsilon, \eta, \zeta) \cap \mathcal{H}(\omega; \epsilon, \eta, \theta) \neq \emptyset$. But now, by Lemma 6.5, we can get between any two triples $\epsilon, \eta, \zeta$ and $\epsilon', \eta', \zeta'$ by a sequence of such moves, replacing one curve at a time. Since $\mathcal{H}(\alpha)$ is a union of such $\mathcal{H}(\omega; \epsilon, \eta, \zeta)$, it follows that $\mathcal{H}(\omega)$ is connected as claimed.

This allows us to construct $\mathcal{G}_s(\Sigma)$ out of $\mathcal{G}_{ss}(\Sigma)$, and so rigidity follows as before. □

Finally we should discuss the cases where $\Sigma \in \{S_{3,4}, S_{4,3}, S_{4,4}\}$. Again, we need to redefine a "surrounding pair":

**Definition.** (When $\Sigma \in \{S_{3,4}, S_{4,3}, S_{4,4}\}$.) A *surrounding pair* is a pair $\alpha, \beta \in C_{ss}(\Sigma)$ with $B(\alpha), B(\beta)$ each either an $S_{0,4}$ or an $S_{1,3}$.

Given this, we define "surrounding triple" as usual.

Note that $B(\alpha)$ is now determined by a curve which is either a component of $\partial\Sigma \setminus B(\omega)$, or a curve $\epsilon \in C_{ss}(\Sigma)$ with $B(\epsilon)$ an $S_{1,1}$ disjoint from $B(\omega)$, together (in either case) with an arc, $a$, from $B(\omega)$ to $\epsilon$. Thus, $B(\alpha)$ is a regular neighbourhood of $B(\omega) \cup a \cup \epsilon$ or of $B(\omega) \cup a \cup B(\epsilon)$, respectively. Surrounding pairs and triples then arise from disjoint curves and arcs similarly as before.

We can define $\mathcal{H}(\omega; \epsilon, \eta, \zeta)$ similarly as in the previous case, where $\epsilon, \eta, \zeta$ are disjoint curves, each either a boundary curve or in $C_{ss}(\Sigma)$ as above. The same argument shows that the graph is connected.

Moreover, we claim we can get between any two such triples, $\epsilon, \eta, \zeta$ and $\epsilon', \eta', \zeta'$, replacing each curve at time by a disjoint curve. In the case where $p(\Sigma) \geq 4$, there are at least two components of $\partial\Sigma \setminus B(\omega)$, so this follows easily applying Lemma 6.4 to $\Sigma \setminus B(\omega)$. If $p(\Sigma) = 3$, then $g(\Sigma \setminus B(\omega)) = g(\Sigma) \geq 4$, and so the statement follows since $\mathcal{S}_2(\Sigma \setminus B(\omega))$ is connected by Lemma 6.5.

The argument can now be completed as before, proving Theorem 1.1 in all cases except $S_{0,8}$.

## 7. The 8-holed sphere

In this section we outline the proof of rigidity for $\mathcal{G}_{ss}(S_{0,8})$. The argument is essentially the same as for $S_{0,7}$, except we need to start by finding a different rigid subgraph, in order to recognise pairs of curves which intersect exactly twice. We will revert to thinking of $S_{0,8}$ as $S \setminus \Pi$, where $\Pi$ is a subset of the 2-sphere, $S$, with $|\Pi| = 8$.

Let $\Delta$ be the graph obtained by adding the four longest diagonal edges to an octagon. More formally, we write $V(\Delta) = \{v_1, \ldots, v_8\}$, where $v_i$ is deemed adjacent to $v_j$ whenever $|i - j| = 1$ or $|i - j| = 4$ (taking indices mod 8).

One can realise $\Delta$ as follows. Recall from Section 3, that $\Theta(\Psi)$ is the graph whose vertex set consists 3-sets in $\Psi$ and where two such sets are deemed adjacent if they are disjoint subsets of $\Psi$. If $\Psi = \{1, \ldots, 8\}$, and $P_i = \{i - 3, i, i + 3\}$, then the full subgraph of $\Theta(\Psi)$ with vertex set $\{P_1, \ldots, P_8\}$ is isomorphic to $\Delta$.

In fact, we claim that all copies of $\Delta$ in $\Theta(\Psi)$ arise in this way.

First note that there are no 3-cycles in $\Theta(\Psi)$, and so any 5-cycle in $\Theta(\Psi)$ is isometrically embedded. Now any two vertices of $\Delta$ lie in a

5-cycle in $\Delta$. It follows that any map of $\Delta$ into $\Theta(\Psi)$ sending edges to edges must be injective.

Suppose that $[v_i \mapsto P_i]$ is any embedding of $\Delta$ into $\Theta(\Psi)$. Now $v_1, v_2, v_6, v_5$ is a 4-cycle in $\Delta$, and so $P_1, P_2, P_6, P_5$ is a 4-cycle in $\Theta(\Psi)$. Let $P_{16} = P_1 \cup P_6$ and $P_{52} = P_5 \cup P_2$. Then $P_{16} \cap P_{52} = \emptyset$, and so $|P_{16}| = |P_{52}| = 4$. In other words, $\{P_{16}, P_{52}\}$ is a partition of $\Psi$ into two 4-sets. The same holds for $\{P_{63}, P_{27}\}$, $\{P_{38}, P_{74}\}$ and $\{P_{85}, P_{41}\}$. Now, $P_{41} \neq P_{16}$. (Otherwise, we would have $P_{41} = P_{16}$ and $P_{85} = P_{52}$, so $P_1 \cap P_2 \subseteq P_{41} \cap P_{52} = \emptyset$. But also $P_3 \cap P_2 = P_3 \cap P_1 = \emptyset$, giving a contradiction.) Similarly, since the $P_i$ are all distinct, one easily sees that $\{P_{16}, P_{52}\} \neq \{P_{38}, P_{74}\}$. It follows that $\{P_{16} \cap P_{38}, P_{16} \cap P_{74}, P_{52} \cap P_{38}, P_{52} \cap P_{74}\}$ is a partition of $\Psi$ into four 2-sets. The same holds for $\{P_{63} \cap P_{85}, P_{63} \cap P_{41}, P_{27} \cap P_{85}, P_{27} \cap P_{41}\}$. From this information, one can easily find a permutation of $\Psi$ so that each $P_i = \{i-3, i, i+3\}$ as in our example. This shows:

**Lemma 7.1.** *There is exactly one embedded copy of $\Delta$ in $\Theta(\Psi)$ up to the action of* $\mathrm{Sym}(\Psi)$.

We now move on to consider $\mathcal{G}_{ss} = \mathcal{G}_{ss}(S_{0,8})$. We set $\Pi = \Psi$. If $\gamma \in C_1 = C_1(S_{0,8})$, write $B(\gamma)$ for the disc with $|\Pi \cap B(\gamma)| = 3$. Let $\pi(\gamma) = \Pi \cap B(\gamma)$. Thus, $\pi$ maps $\mathcal{G}(S_{0,8}, C_1)$ to $\Theta(\Pi)$ sending edges to edges.

We note that $\mathcal{G}(S_{0,8}, C_1) \subseteq \mathcal{G}_{ss}(\Sigma)$ contains an embedded copy of $\Delta$ constructed as follows. Let $\lambda \subseteq S$ be an embedded circle with $\Pi \subseteq \lambda$. We label the elements of $\Pi$ so that the cyclic order induced from $\lambda$ is given by $p_1, p_4, p_7, p_2, p_5, p_8, p_3, p_6$. Let $l_{i,i+3} \subseteq \lambda$ be the segment from $p_i$ to $p_{i+3}$. Let $B_i$ be a regular neighbourhood of the arc $l_{i-3,i} \cup l_{i,i+3}$ and let $\gamma_i = \partial B_i$. The map $[v_i \mapsto \gamma_i]$ now gives an embedding of $\Delta$ into $\mathcal{G}(S_{0,8}, C_1)$.

We claim:

**Lemma 7.2.** *There is exactly one embedded copy of $\Delta$ in $\mathcal{G}(S_{0,8}, C_1)$ up to the action of* $\mathrm{Map}(S_{0,8})$.

Let $\Delta \subseteq \mathcal{G}(S_{0,8}, C_1)$ be such a copy. After composing with $\pi$, we get a map of $\Delta$ into $\Theta(\Pi)$ sending edges to edges, which as we have already noted, must also be an embedding. By Lemma 7.1, we can now label the elements of $\Pi$ as $\{p_1, \ldots, p_8\}$ so that $\pi(\gamma_i) = \{p_{i-3}, p_i, p_{i+3}\}$ for all $i$. (In other words, as in the given example.) Write $B_i = B(\gamma_i)$.

Now $\gamma_1, \gamma_2, \gamma_6, \gamma_5$ is a square in $\Delta$, and so $(B_1 \cup B_6) \cap (B_5 \cup B_2) = \emptyset$. Also $|(B_1 \cup B_6) \cap \Pi| = |(B_5 \cup B_2) \cap \Pi| = 4$. It follows that we can find disjoint discs, $B_{16}, B_{52} \subseteq S$ with $B_1 \cup B_6 \subseteq B_{16}$ and $B_5 \cup B_2 \subseteq B_{52}$. Note that $\partial B_{16}$ and $\partial B_{52}$ are homotopic in $S \setminus \Pi$, and so determine a

curve, $\gamma_{16} = \gamma_{52}$. We have similar pairs of discs, $\{B_{63}, B_{27}\}$, $\{B_{38}, B_{74}\}$ and $\{B_{85}, B_{41}\}$. Note that we have similar curves, $\gamma_{ij}$, defined whenever $|i - j|$ is 3 or 5.

Now consider the curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Let $A = S \setminus (B_1 \cup B_2)$. This is an annulus with $A \cap \Pi = \{p_3, p_8\}$. Now $B_3 \cap B_2 = \emptyset$, $B_3 \cap B_1 \neq \emptyset$ (since $p_6 \in B_3 \cap B_1$), and $p_3, p_8 \in B_3 \cap A$. Thus, we can find (disjoint) arcs, $e, f$ in $B_3 \cap A$, respectively connecting $p_3$ and $p_8$ to $\partial B_1$.

Let $D$ be (the closure of) the component of $B_4 \setminus B_1$ containing $p_7$. Now $B_4 \cup B_1 \subseteq B_{41}$ and $(B_{41} \setminus B_4) \cap \Pi = \{p_7\}$. It follows (cf. Lemma 3.4) that $D \cap B_1$ is a single arc, $b$, say. Thus, $\partial D = b \cup d$, where $d \subseteq \gamma_4$ is an arc with endpoints $d \cap B_1 = d \cap b$.

Now, $d \cap (e \cup f) \subseteq B_4 \cap B_3 = \emptyset$. It follows that any arc of $d \cap A$ which does not include an endpoint could be homotoped into $B_2$. It follows that $d \cap B_2$ must consist of a single arc, and so $|d \cap B_2| = 2$. Now $R = B_1 \cup D$ is a disc with $R \cap \Pi = B_{41} \cap \Pi$, and so $\partial R$ is homotopic in $S \setminus \Pi$ to $\gamma_{41} = \gamma_{52}$. Also $|\partial R \cap \gamma_2| = 2$, and so $\iota(\gamma_{41}, \gamma_2) = 2$.

By symmetry, it follows that each curve $\gamma_i$ intersects each curve of the form $\gamma_{jk}$ either 0 or 2 times.

It is now fairly straightforward to see that $\{\gamma_1, \ldots, \gamma_8\}$ must be precisely the set of curves described in our example. For example, note that $B_{41} \cap \Pi = \{p_7, p_4, p_1, p_6\}$, and $B_{41} \cap \gamma$ is an arc which cuts off a disc, namely $B_{41} \cap B_2$, containing $p_7$. Now $B_1 \cap \Pi = \{p_4, p_1, p_6\}$, and so $\gamma_1 = \partial B_1$ is homotopic to $\partial(B_{41} \setminus B_2)$. Similarly, $\gamma_4$ is homotopic to $\partial(B_{41} \setminus B_3)$. It follows that $B_1 \cap B_4$ is a disc containing $p_1, p_4$. One can now proceed to show that any pair of the curves $\gamma_i$ intersect at most twice, and that this determines them completely.

This proves Lemma 7.2.

The remainder of the proof of rigidity is now essentially identical to that for $S_{0,7}$. We define surrounding pairs in the same way. A surrounding pair, $\alpha, \beta$, (as defined in Section 2) can be recognised by the fact that $\alpha, \beta \in C_1(S_{0,8})$, there is a curve $\gamma \in C_{ss}(S_{0,8}) \setminus C_1(S_{0,8})$ disjoint from both $\alpha$ and $\beta$, and $\alpha, \beta$ are vertices of some embedded copy of $\Delta$ in $\mathcal{G}(S_{0,8}, C_1)$. (Note that $\mathcal{G}(S_{0,8}, C_1)$ can be constructed from $\mathcal{G}_{ss}(S_{0,8})$.) We now proceed as before.

This shows that $\mathcal{G}_{ss}(S_{0,8})$ is rigid, completing the proof of Theorem 1.1.

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